

Journal of Financial and Actuarial Mathematics and Management

A joint edition of the Institute of Information Sciences and Institute of Social Sciences of Perun Academy
Publishing Company: PSP Publishing
ISSN: 2367-5810
Issue 1, Year I (2013)
Founding Editor and Editor-in-Chief: Pavel Stoynov

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On some approximations used in the risk process of an insurance company

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Abstract.
In an insurance company, the risk process estimation and the estimation of the ruin probability are important concerns for an actuary: for researchers, at the theoretical level, and for the management of the company, as these influence the insurer strategy. We consider the evolution over an extended period of time of an insurer surplus process. In this paper, we present some methods of estimating of the ruin probability. We discuss the approximations of ruin probability with respect to: the parameters of the individual claim distribution, the load factor of premiums, and the intensity parameter of the number of claims process. We analyze the model where the premiums are computed on the basis of the mean value principle. We give numerical illustration.

JEL codes: C020, G220, G320
Key Words: ruin probability, risk process, adjustment coefficient.

*This article is presented at the First international conference "Financial and Actuarial mathematics – FAM"-2008, Sofia, Bulgaria

Introduction

The mathematical model of an insurance risk business consists (is composed) of the following objects:

a) A sequence $\{X_i\}_{i=1,2,3,...}$ of independent and identically distributed random variables (i.i.d.r.v.), having the common cumulative distribution function F. X_i is the cost of the i-th individual claim.

b) A stochastic process $N=\{N(t);t\geq 0\}$. $N(t)$ is the number of claims paid by the company in the time interval $[0,t]$. The counting process N and the sequence $\{X_i\}_i$ are independent.

The total amount of claims paid by the company in the time interval $[0,t]$ is $D(t)=\sum_{i=1}^{N(t)} X_i$, $\{D(t);t\geq 0\}$ being the claim process. The risk process $Y=\{Y(t);t\geq 0\}$ is defined by $Y(t)=c\cdot t-D(t)$, where c is a positive real constant number corresponding to the average premium income per unit of time. We shall consider that the process N is a homogeneous Poisson process with intensity λ and that we will use the mean value principle in order to compute the net premiums, thus $c=(1+\theta)\cdot\lambda\cdot m_1$, where θ is the relative safety load factor and m_1 is the expected value of the individual claim or the expected cost of a claim. We denote $m_k=E\left[X_i^k\right], k=1,2,3,\ldots$. We shall denote by r – the initial capital, $C(t)$ - the capital of the

company at moment t, hence $r = C(0)$ and $C(t) = r + Y(t)$. We define the ruin as being the situation when the capital of the company takes a negative value. The ruin moment, τ, is defined as $\tau = \inf\{t \geq 0 \,|\, C(t) < 0\}$.

Let $h(r) = \int_0^{\infty} \left(e^{r \cdot x} - 1\right) dF(x)$ and $g(\theta) = (1+\theta) \cdot \lambda \cdot m_1$. We denote by $\Psi_n(r, \theta)$ the ruin probability up to moment n and by $\Psi(r, \theta)$ the ruin probability on an infinite time horizon, so

$$\Psi_n(r, \theta) = P\left(\tau < n \,|\, C(0) = r,\, g(\theta) = c\right),$$
$$\Psi(r, \theta) = P\left(\tau < \infty \,|\, C(0) = r,\, g(\theta) = c\right)$$

and

$$\Psi(r, \theta) = \lim_{n \to \infty} \Psi_n(r, \theta).$$

The adjustment coefficient (or Lundberg exponent) R is the positive solution of the equation $\lambda \cdot h(r) - c \cdot r = 0$.

A well-known result states that: if the adjustment coefficient R exists, the ruin probability is

$$\Psi(r, \theta) = e^{-R \cdot r} \cdot \left(E\left[e^{R \cdot S(\tau)}\right]\right)^{-1}, \qquad (1)$$

where $S(\tau) = (-C(\tau) | \tau < \infty))$ represents the severity of the loss at the moment of ruin.

In case the individual claim follows a negative exponential distribution, $X_i \in Exp(\alpha), \alpha > 0$ then

$$\Psi(r, \theta) = \frac{\lambda}{\alpha \cdot g(\theta)} \cdot e^{-\left(\alpha - \frac{\lambda}{g(\theta)}\right) \cdot r} \qquad (2)$$

The integrated tail distribution is $F_I(z) = \frac{1}{m_1} \cdot \int_0^z (1 - F(x))\, dx$. It is known the the Beekman's convolution (or Pollaczek-Khinchine) formula:

$$\Psi(r, \theta) = \frac{\theta}{1+\theta} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n \cdot \tilde{F}_I^{n*}(r) \qquad (3)$$

where $\tilde{F}_I(r) = 1 - F_I(r)$. If F is the cumulative distribution function for a negative exponential distributed loss, then $F_I = F$.

Some approximations of the ruin probability

De Vylder (1978) proposed an approximation based on the idea to replace the risk process Y by a risk process $\tilde{Y}$ with exponential distributed claims such that $E\left[\tilde{Y}^k(t)\right] = E\left[Y^k(t)\right], k = 1, 2, 3.$ The risk process $\tilde{Y}$ is determined by the parameters $\left(\tilde{\lambda}, \tilde{\theta}, \tilde{\alpha}\right)$. Since $\ln z = \ln r + i \cdot \theta$, where $z = r \cdot e^{i \cdot \theta}$, we have $\ln E\left[e^{i \cdot \nu \cdot Y(t)}\right] = t \cdot \left(i \cdot \nu \cdot \theta \cdot \lambda \cdot m_1 - \frac{\nu^2}{2} \cdot \lambda \cdot m_2 + i \cdot \frac{\nu^3}{3!} \cdot \lambda \cdot m_3 + 0\left(\nu^3\right)\right)$ and we get $\tilde{m}_1 = \frac{1}{\tilde{\alpha}} = \frac{m_3}{3 \cdot m_2}$, $\tilde{\theta} = \frac{2}{3} \cdot \frac{m_1 \cdot m_3}{m_2^2} \cdot \theta$ and $\tilde{\lambda} = \frac{9}{2} \cdot \frac{m_2^3}{m_3^2} \cdot \lambda.$ Thus, we obtain the approximation

$$\Psi(r,\theta)\approx\Psi_{DV}(r,\tilde{\theta})=\frac{1}{1+\tilde{\theta}}\cdot e^{-\frac{\tilde{\theta}\cdot\tilde{\alpha}}{1+\tilde{\theta}}\cdot r} \tag{4}$$

$$\Psi_{DV}(r,\theta)=\frac{3\cdot m_2^2}{3\cdot m_2^2+2\cdot m_1\cdot m_3\cdot\theta}\cdot e^{-\frac{6\cdot m_1\cdot m_2\cdot\theta\cdot r}{3\cdot m_2^2+2\cdot m_1\cdot m_3\cdot\theta}} \tag{5}$$

If $X:\begin{pmatrix}n_i\\p_i\end{pmatrix}_{i\in I\subset\mathbf{N}}$, $p_i>0,\sum_{i\in I}p_i=1$, we introduce $b_i=\frac{n_i\cdot p_i}{m_1}$, $a_i=\frac{n_i^2\cdot p_i}{m_2}$, $X':\begin{pmatrix}n_i\\a_i\end{pmatrix}_{i\in I}$, $X'':\begin{pmatrix}n_i\\b_i\end{pmatrix}_{i\in I}$ and we obtain $\tilde{m}_1=\frac{1}{3}\cdot E(X')$, $\tilde{\theta}=\frac{2}{3}\cdot\frac{E(X')}{E(X'')}\cdot\theta$, $\tilde{\lambda}=\frac{9}{2}\cdot\frac{E(X)\cdot E(X'')}{E(X')}\cdot\lambda$

and

$$\Psi_{DV}(r,\theta)=\frac{3\cdot E(X'')}{3\cdot E(X'')+2\cdot E(X')\cdot\theta}\cdot e^{-\frac{6\cdot\theta\cdot r}{3\cdot E(X'')+2\cdot E(X')\cdot\theta}} \tag{6}$$

The most famous approximation is the Cramer-Lundberg approximation:

$$\Psi(r,\theta)\approx\Psi_{CL}(r,\theta)=\frac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R)-c}\cdot e^{-R\cdot r} \tag{7}$$

Let SE be the class of subexponential distributions, i.e. $F\in SE$ if $\lim_{x\to\infty}\frac{\tilde{F}^{2*}(x)}{\tilde{F}(x)}=2$. It is shown by Embrechts and Veroverbeke (1982), that, if $F_I\in SE.$, then

$$\Psi(r,\theta)=\frac{1}{1+\theta}\cdot\tilde{F}_I(r) \tag{8}$$

We showed that if $F_I\in SE$ then

$$\lim_{r\to\infty}\frac{\Psi(r,\theta)}{1-F_I(r)}=\frac{\lambda\cdot m_1}{g(\theta)-\lambda\cdot m_1} \tag{9}$$

Let $H(r)=P\left(-\inf_{t\ge0}Y(t)\le r\;\middle|\;-\inf_{t\ge0}Y(t)>0\right)$.

We have $\Psi(r,\theta)=\frac{1}{1+\theta}\cdot(1-H(r))$, $\mu_H=\frac{(1+\theta)\cdot m_2}{2\cdot\theta\cdot m_1}$ and $\sigma_H^2=\frac{(1+\theta)\cdot m_2}{2\cdot\theta\cdot m_1}\left(\frac{2m_3}{3m_2}+\frac{(1-\theta)m_2}{2\cdot\theta\cdot m_1}\right)$, where μ_H and σ_H^2 are the mean and the variance corresponding to H. The idea of Beekman-Bowers approximation is to replace $H(r)$ by a Γ-distribution function $G(r)$, such that the two first moments of H and G should match. In this way, it is obtained the approximation formula

$$\Psi(r,\theta)\approx\Psi_{\mathrm{BB}}(r,\theta)=\frac{1}{(1+\theta)\cdot\Gamma(a)}\int_{br}^{\infty}x^{a-1}\cdot e^{-x}dx \tag{10}$$

where $a=\frac{3(1+\theta)\cdot m_2^2}{3m_2^2+\theta\left(4m_1m_3-3m_2^2\right)}$ and $b=\frac{6m_1m_2\theta}{3m_2^2+\theta\left(4m_1m_3-3m_2^2\right)}$.

In the case of exponentially distributed claims we have $a=1$, $b=\frac{\alpha\theta}{1+\theta}$ and $\Psi_{\mathrm{BB}}(r,\theta)=\frac{1}{1+\theta}\cdot e^{-\frac{\alpha\theta r}{1+\theta}}=\Psi(r,\theta)$.

The simplest approximation, which only depends on some moments of F, seems to be the diffusion approximation:

$$\Psi(r,\theta) \approx \Psi_{\mathrm{D}}(r,\theta) = e^{-2\cdot\frac{m_1}{m_2}\cdot\theta\cdot r} \qquad (11)$$

As the Lundberg exponent R is small for small values of θ we have

$$h(R) = m_1 \cdot R + \frac{m_2}{2}\cdot R^2 + o\left(R^3\right)$$

which leads to

$$R = 2\frac{m_1}{m_2}\cdot\theta + o\left(\theta^2\right)$$

and

$$\Psi_{\mathrm{CL}}(r,\theta) = \frac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R) - c}\cdot e^{-R\cdot r} = \frac{1}{1+o\left(\theta^2\right)}\cdot e^{-2\frac{m_1}{m_2}\cdot\theta\cdot r + o\left(\theta^2\right)}.$$

Thus the diffusion approximation may be regarded as a simplified Cramer-Lundberg approximation. Since the diffusion approximation is not very accurate, Grandell (2000) proposed to use for $h(r)$ three moments.

Thus $R = 2\frac{m_1}{m_2}\cdot\theta - \frac{4m_1^2 m_3\theta^2}{3m_2^3} + o\left(\theta^3\right)$, $\frac{\theta\cdot\lambda\cdot m_1}{\lambda\cdot h'(R) - c} = \frac{3m_2^2}{3m_2^2 + 2m_1 m_3\theta} + o\left(\theta^2\right)$ and

$$\Psi_{\mathrm{G}}(r,\theta) = \frac{3m_2^2}{3m_2^2 + 2m_1 m_3\theta}\cdot e^{-\left(2\frac{m_1}{m_2}\cdot\theta - \frac{4m_1^2 m_3\theta^2}{3m_2^3}\right)\cdot r} \qquad (12)$$

Since $2\frac{m_1}{m_2}\cdot\theta - \frac{4m_1^2 m_3\theta^2}{3m_2^3} = \frac{6m_1 m_2\theta}{3m_2^2 + 2m_1 m_3\theta} + o\left(\theta^2\right)$, the De Vylder approximation may be regarded as a simplified Grandell approximation.

Another approximation is obtained using Renyi's theorem about the p-thinning of the point process. Thus

$$\Psi_{\mathrm{R}}(r,\theta) = \frac{1}{1+\theta}\cdot e^{-\frac{2m_1\theta r}{m_2\cdot(1+\theta)}} \qquad (13)$$

Kalashnikov (1997) showed that $\sup_r \left|\Psi_{\mathrm{R}}(r,\theta) - \Psi(r,\theta)\right| \le \frac{4m_1 m_3\theta}{3m_2^2(1+\theta)}$ for all $\theta > 0$.

Numerical results and conclusions

In this section, our purpose is to compare the different approximations listed above, through a numerical example. We have to deal with the absolute error (δ) and the relative error (ε). Thus, for approximation Ψ_{A} of Ψ we have $\delta_{\mathrm{A}}(r,\theta) = \left|\Psi_{\mathrm{A}}(r,\theta) - \Psi(r,\theta)\right|$ and $\varepsilon_{\mathrm{A}}(r,\theta) = \frac{\delta_{\mathrm{A}}(r,\theta)}{\Psi(r,\theta)}$. In order to compare an approximation Ψ_{A} with an other approximation Ψ_{B}, we will use $\delta_{\mathrm{AB}}(r,\theta) = \left|\Psi_{\mathrm{A}}(r,\theta) - \Psi_{\mathrm{B}}(r,\theta)\right|$ and $\varepsilon_{\mathrm{AB}}(r,\theta) = \frac{\delta_{\mathrm{AB}}(r,\theta)}{\Psi_{\mathrm{A}}(r,\theta)}$.

Let $X:\begin{pmatrix}1 & 5\\ 0.875 & 0.125\end{pmatrix}$ be a discrete random variable describing a claim (a loss) which takes the value of 1 monetary unit with a high probability, and a large value of 5 monetary units (in the case of a natural disaster, for example) with a relatively low probability. In the following, we list the approximations of the ruin probability (ARP):

Table 1

ARP for $\theta = 0.2$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.732078	0.733542	0.735414	0.860708
5	0.445179	0.444494	0.446051	0.472367
10	0.239060	0.238377	0.238754	0.223130
20	0.068937	0.068845	0.068480	0.049787
30	0.019879	0.019917	0.019598	0.011109
40	0.005732	0.005767	0.005615	0.002479
50	0.001653	0.001670	0.001609	0.000553

Table 2

ARP for $\theta = 0.3$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.643143	0.644393	0.646979	0.798516
5	0.323441	0.322402	0.323761	0.324652
10	0.136979	0.136533	0.136268	0.105399
20	0.024568	0.024610	0.024140	0.011109
30	0.004406	0.004447	0.004276	0.001171
40	0.000790	0.000805	0.000758	0.000123
50	0.000142	0.000146	0.000134	0.000013

Table 3

ARP for $\theta = 0.5$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.515174	0.516071	0.519201	0.687289
5	0.191486	0.190638	0.191003	0.153355
10	0.055573	0.055507	0.054723	0.023518
20	0.004681	0.004740	0.004492	0.000553
30	0.000394	0.000407	0.000369	0.000013
40	0.000033	0.000035	0.000030	$0.3\cdot 10^{-6}$
50	0.000003	0.000003	0.000002	$0.7\cdot 10^{-8}$

Table 4

ARP for $\theta = 0.8$

r	Ψ_{DV}	Ψ_{BB}	Ψ_{R}	Ψ_{D}
1	0.394417	0.394444	0.398073	0.548812
5	0.105883	0.105335	0.104931	0.049787
10	0.020461	0.020516	0.019819	0.002479
20	0.000764	0.000789	0.000707	0.000006
30	0.000029	0.000030	0.000025	$0.15 \cdot 10^{-7}$
40	0.000001	0.000001	$0.9 \cdot 10^{-6}$	$0.38 \cdot 10^{-10}$
50	$0.4 \cdot 10^{-7}$	$0.5 \cdot 10^{-7}$	$0.3 \cdot 10^{-7}$	$0.9 \cdot 10^{-13}$

We notice that the approximations Ψ_{DV}, Ψ_{BB} and Ψ_{R} are better as they have maximum absolute errors up to 0.0015 for $\theta = 0.2$, up to 0.0013 for $\theta = 0.3$, up to 0.0009 for $\theta = 0.5$ and maximum relative errors up to 2.7% for $\theta = 0.2$, up to 5.5% for $\theta = 0.3$, etc. The diffusion approximation is not very accurate because its relative errors are very high, up to 99% for $\theta = 0.5$

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Some applications of sums of random variable in non-life insurance

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Abstract.
In the non-life insurance business, an actuary faces the problem of determining the distribution function of a sum of random variables which are not necessarily independent, like aggregate claims of an insurance portfolio. The paper points out some applications of approximating such sums when the individual distribution functions of the terms are known, but their dependence unknown.
Key words: comonotonicity, lognormal distribution, insurance premium principle, stop-loss premium.

*This article is presented at the First international conference "Financial and Actuarial mathematics – FAM"-2008, Sofia, Bulgaria

Introduction

In the actuarial domain, both life and non-life insurance, an actuary faces the problem of determining the distribution of a sum of random variables.

We deal with such sums when we consider: the total claim associated to a portfolio of risks corresponding to a certain limited interval of time, regarding damage compensations, or the accumulated fund at the end of an extended period of time (an investment problem), or the present value of a sequence of deterministic payments at successive moments of time which are meant to cover some future compulsory payments.

The assumption that in a portfolio, the risks, described as random variables, are independent is convenient (suitable) for the calculus. In reality, the random variables involved are dependent, this fact making the computations regarding the distribution of the sum quite complicated. Thus, a new problem appears: the one concerning how to find the distribution of the sum of non-independent random variables. In the case we know the individual distributions of each term, but not the stochastic dependence between them, one may determine approximations for such a sum. From an actuary point of view, these approximations can be "pessimistic" and "optimistic", reflecting his behavior characterized by aversion to risk. His preferences are modeled by convex ordering.

A special class of random variables allowing such approximations is the class of comonotonic random variables. We will introduce the notion of comonotony, we will present some theoretical results which will lead us to the conclusion that the initial aggregate loss of a portfolio is dominated by the aggregate comonotonic loss, according to the convex order. We will give a numerical illustration for a portfolio of uniformly distributed risk variables.

Theoretical results

The whole information on a random variable X can be found in its cumulative distribution function F_X. For an insurance company, the random variable could represent the risk associated to an insurance policy. An actuary uses various mathematical and statistical techniques in order to determine (find) the cumulative distribution function. Often, in the non-life insurance, for example, he will deal with random variables like the aggregate loss $S = \sum_{i=1}^{n} X_i$, where the terms $X_1,...,X_n$ are not mutually independent, and the distribution of the random vector $\underline{X} \overset{not}{=} (X_1,...,X_n)$ is not completely specified or much too complicated, knowing only the marginal distributions of the random variables X_i.

The insurance company is interested to propose a cautious strategy, meaning a dependency structure for a random variable $\underline{X}$, dependency which is meant to provide a larger or less favorable aggregate loss in the sense of convex order. We will use the convex order for ordering risks because, in the utility theory, for decidents with risk aversion that have to choose between risks having the same expected value, their preferences are modeled by it.

One can prove that for a random vector with given marginals, the largest sum (or aggregate loss) in the convex order sense can be obtained in the case the random vector $(X_1,...,X_n)$ has comonotonic distribution, i.e. any two possible realizations $(x_1,...,x_n)$ and $(y_1,...,y_n)$ of the random vector $(X_1,...,X_n)$ are componentwise ordered. If $\underline{x} = (x_1,...,x_n)$ and $\underline{y} = (y_1,...,y_n)$ are two vectors in the n-dimensional $\mathbf{R}^n$ real vector space, the componentwise ordering is defined as: $\underline{x} \le \underline{y} \Leftrightarrow x_i \le y_i, \forall i = \overline{1,n}$.

Definition. The set $A \subseteq R^n$ is called a comonotonic set if for any vectors $\underline{x}, \underline{y} \in A$, then $\underline{x} \le \underline{y}$ or $\underline{y} \le \underline{x}$.

The support of the random vector $\underline{X} = (X_1,...,X_n)$ is the smallest set $A \subseteq R^n$ with the property $P(\underline{X} \in A) = 1$.

Definition. A random vector $\underline{X} = (X_1,...,X_n)$ is comonotonic if it has a comonotonic support.

By definition, if $\underline{x}$ and $\underline{y}$ are two elements of the comonotonic support of the random vector $\underline{X}$, i.e. two possible outcomes of $\underline{X}$, then they must be ordered on components. This explains the use of the word comonotony (common monotony). A larger value of one of the components X_j of the random vector $\underline{X}$ implies a larger value of any other component X_k.

The concept of comonotony was introduced by Schmeidler (Schmeidler, D. (1986). „Integral representation without additivity", Proceedings of the American Mathematical Society 97, 255-261), and by Yaari (Yaari, M.E. (1987). „The dual theory of choice under risk", Econometrica 55, 95-115) and by Roell (Roell, A. (1987). „Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty", The Economic Journal 97 (Conference 1987), 143-159).

The cumulative distribution function of the random variable X is $F_X : R \to [0,1]$, $F_X(x) = P(X \le x)$ an increasing and continuous to the right function with $F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0$ and $F_X(+\infty) = \lim_{x \to +\infty} F_X(x) = 1$.

The inverse of the distribution function is $F_X^{-1} : [0,1] \to R$ an increasing and continuous to the left function defined by $F_X^{-1}(p) = \inf\{x \in R | F_X(x) \ge p\}, p \in [0,1]$, with the convention $\inf \phi = +\infty$. From the definition it follows that $\forall x \in \mathbf{R}$ and $\forall p \in [0,1]$, then $F_X^{-1}(p) \le x \Leftrightarrow p \le F_X(x)$.

We denote by U the random variable uniformly distributed over the interval (0,1), $U \in Uniform(0,1)$, $f_U(u) = 1, \forall u \in (0,1)$, $F_U(u) = P(U \le u) = u, \forall u \in (0,1)$ and $F_U^{-1}(p) = p, \forall p \in (0,1)$.

The following theorem gives equivalent characterizations of a comonotonic random vector.

Theorem Let $\underline{X} = (X_1,..., X_n)$ be a random vector. The following are equivalent:

(1') $\underline{X}$ is a comonotonic random vector;

(1) $\underline{X}$ had a comonotonic support;

(2) For any $\underline{x} = (x_1,..., x_n)$, $F_{\underline{X}}(\underline{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2),..., F_{X_n}(x_n)\}$;

(3) For $U \in Uniform(0,1)$, we have $\underline{X} \overset{(R)}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U),..., F_{X_n}^{-1}(U))$, $\overset{(R)}{=}$ means equality in distribution;

(4) There exist a random variable Z and the increasing functions $f_i, i = \overline{1,n}$, such that $\underline{X} \overset{(R)}{=} (f_1(Z), f_2(Z),..., f_n(Z))$.

Proposition $X \overset{(R)}{=} F_X^{-1}(U)$, where U is the random variable uniformly distributed on (0,1), $U \in Uniform(0,1)$, and $\overset{(R)}{=}$ means equality in distribution.

Theorem Let X and $g(X)$ two random variables and let $p \in (0,1)$. If g is an increasing and continuous to the left function, then $F_{g(X)}^{-1}(p) = g(F_X^{-1}(p))$;

In the following, for any random vector $(X_1,..., X_n)$, we will denote by $(X_1^c,..., X_n^c)$ the comonotonic random vector having the same marginal distributions as the random vector $(X_1,..., X_n)$. For a random vector $\underline{X} = (X_1,..., X_n)$, a realization of the corresponding comonotonic random vector $\underline{X}^c = (X_1^c,..., X_n^c)$ lies, with probability 1, within the set $\{(F_{X_1^c}^{-1}(p),..., F_{X_n^c}^{-1}(p)) | p \in (0,1)\}$, or $\{(F_{X_1}^{-1}(p),..., F_{X_n}^{-1}(p)) | p \in (0,1)\}$ as

$X_1 \overset{(R)}{=} X_1^c,..., X_n \overset{(R)}{=} X_n^c$, where we used the fact that

$\underline{X}^c = (X_1^c,..., X_n^c) \overset{(R)}{=} (F_{X_1^c}^{-1}(U),..., F_{X_n^c}^{-1}(U))$, with $U \in Uniform(0,1)$.

For the comonotonic random vector $\left(X_1^c,...,X_n^c\right)$, we will denote $S^c = X_1^c + ... + X_n^c$ the sum of its components. We will prove that approximating the distribution of the aggregate loss $S = X_1 + ... + X_n$ by the distribution of the comonotonic sum S^c is a prudent strategy, in the sense of the convex order, i.e. $S \le_{cx} S^c$.

Stochastic order: The concept of stochastic dominance and stop-loss order have been introduced in the actuarial science because the essence of the actuary profeesion is to compare risks.

Definition: Let X and Y be two random variables describing the losses or the risks. About them we make the assumption that they have finite means.

X is said to precede Y in the stochastic dominance sense

$X \le_{st} Y \Leftrightarrow F_X(x) \ge F_Y(x), \forall x \in R.$

X is said to precede Y in the stop-loss order sense if and only if X has lower stop-loss premium than Y, i.e.

$X \le_{sl} Y \Leftrightarrow E\left[(X-d)_+\right] \le E\left[(Y-d)_+\right], \forall d \in \mathbf{R}.$

X is said to precede Y in the convex order sense

$X \le_{cx} Y \Leftrightarrow E\left[(X-d)_+\right] \le E\left[(Y-d)_+\right], \forall d \in \mathbf{R}$ and, in addition $E[X] = E[Y]$.

The following theorem gives the inverse of the cumulative distribution function for the comonotonic sum.

Theorem The inverse of $F_{S^c}(p)$ verifies $F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \forall p \in (0,1)$.

Proof: Let us consider the random vector $(X_1,...,X_n)$ and let $\left(X_1^c,...,X_n^c\right)$ be the corresponding comonotonic random.

As $\underline{X}^c = \left(X_1^c,...,X_n^c\right) \overset{(R)}{=} \left(F_{X_1^c}^{-1}(U),...,F_{X_n^c}^{-1}(U)\right)$,

then $S^c = X_1^c + ... + X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U) + ... + F_{X_n}^{-1}(U)$, for $U \in Uniform(0,1)$.

We introduce the function $g(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u), u \in (0,1)$, which is an increasing and continuous to the left function, $S^c \overset{(R)}{=} g(U)$.

Thus $F_{S^c}^{-1}(p) = F_{g(U)}^{-1}(p) = g\left(F_U^{-1}(p)\right) = g(p), p \in (0,1)$

$\Rightarrow F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), p \in (0,1)$.

Stop-loss cover of the losses

In the insurance business, a widely used practice is to cover only partially the risks or the losses. Among the different schemes dividing the loss between insured-insurer or insurer-reinsurer, one can distinguish that the one which is often used is the stop-loss cover, where a deductible is introduced. We distinguish individual stop-loss cover on components, and respectively, the collective or aggregate stop-loss cover.

Definition. The stop-loss premium associated to the risk X, with deductible $d \in \mathbf{R}$, is defined as $E\left[(X-d)_+\right]$, where $(a-b)_+ = \begin{cases} 0, & a \le b \\ a-b, & a > b \end{cases}$.

The next proposition gives formulas for stop-loss premiums associated to risks which follow normal, lognormal, and uniform distribution laws.

Proposition

1) Let $X \in N(\mu, \sigma^2)$ and $d \in \mathbf{R}$,

then the stop-loss premium is $E\left[(X-d)_+\right] = \mu - d + \sigma \int_{-\infty}^{\frac{d-\mu}{\sigma}} \phi(y)dy$.

2) Let $X \in Lognorma(\mu, \sigma^2)$ and $d > 0$,

then the stop-loss premium is $E\left[\left(X-d\right)_+\right]=e^{\mu+\frac{\sigma^2}{2}}\cdot\left(1-\phi\left(\alpha-\sigma\right)\right)-d\cdot\left(1-\phi\left(\alpha\right)\right)$,

where $\alpha=\frac{\ln d-\mu}{\sigma}$, and $\phi\left(z\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}t^2}dt, z\in\mathbf{R}$ being the cumulative distribution function of the standard normal distribution law $N\left(0,1\right)$.

3) Let $X\in Uniform\left(\alpha,\beta\right)$, $0<\alpha<\beta$ and $d>0$, then

$$E\left[\left(X-d\right)_+\right]=\begin{cases}\frac{\alpha+\beta}{2}-d & ,0<d\le\alpha\\ \frac{\left(\beta-d\right)^2}{2\left(\beta-\alpha\right)} & ,\alpha<d<\beta\\ 0 & ,d\ge\beta\end{cases}.$$

Proof Let X be a random variable with finite expected value and let $d\in\mathbf{R}$.

$\left(X-d\right)_+=\begin{cases}0, & X\le d\\ X-d, & X>d\end{cases}$, $\left(d-X\right)_+=\begin{cases}d-X, & X\le d\\ 0, & X>d\end{cases}$

Integrating by parts, one can show that:

$$E\left[\left(X-d\right)_+\right]=\int_d^{+\infty}\left(x-d\right)dF_X\left(x\right)=$$

$$=-\left(x-d\right)\left(1-F_X\left(x\right)\right)\Big|_d^{+\infty}+\int_d^{+\infty}\left(1-F_X\left(x\right)\right)dx=$$

$$=\int_d^{+\infty}\overline{F_X}\left(x\right)dx,$$

Where $\overline{F_X}\left(x\right)=P\left(X>x\right)=1-P\left(X\le x\right)=1-F_X\left(x\right)$ is the survival function,

We used $\lim_{x\to\infty}\left(x-d\right)\left(1-F_X\left(x\right)\right)=\lim_{x\to\infty}x\left(1-F_X\left(x\right)\right)-\lim_{x\to\infty}d\left(1-F_X\left(x\right)\right)$, $\lim_{x\to\infty}d\left(1-F_X\left(x\right)\right)=0$.

As $E\left[X\right]<\infty$, meaning that the generalized integral $\int_{-\infty}^{+\infty}x\,dF_X\left(x\right)$ is convergent, we have that

$\int_x^{+\infty}t\,dF_X\left(t\right)\xrightarrow[x\to\infty]{}0$ ("the tails" are convergent to 0).

With this observation, we obtain that, for $x\to\infty$, $\left|x\left(1-F_X\left(x\right)\right)\right|=\left|x\right|\left(1-F_X\left(x\right)\right)=x\int_x^{\infty}dF_X\left(t\right)\le\int_x^{\infty}t\,dF_X\left(t\right)\xrightarrow[x\to\infty]{}0$, that is $\lim_{x\to\infty}x\left(1-F_X\left(x\right)\right)=0$.

(Comment: the stop-loss premium with deductible d can be thought as the weight of the upper tail of the distribution of the random variable X, i.e. the area of the domain which lies between the graph of the cumulative distribution function and the constant function 1, from d la $+\infty$.

Analogously,

$$E\left[\left(d-X\right)_+\right]=\int_{-\infty}^{d}\left(d-x\right)dF_X\left(x\right)=$$

$$=\left(d-x\right)F_X\left(x\right)\Big|_{-\infty}^{d}+\int_{-\infty}^{d}F_X\left(x\right)dx=$$

$$=\int_{-\infty}^{d}F_X\left(x\right)dx,$$

with $\lim_{x\to-\infty}\left(d-x\right)F_X\left(x\right)=\lim_{x\to-\infty}d\,F_X\left(x\right)-\lim_{x\to-\infty}x\,F_X\left(x\right)$, $\lim_{x\to-\infty}d\,F_X\left(x\right)=d\cdot0=0$.

As $E[X]<\infty$, then the generalized integral $\int_{-\infty}^{+\infty} x\,dF_X(x)$ is convergent, and we have that $\int_{-\infty}^{x} t\,dF_X(t)\xrightarrow[x\to-\infty]{}0$ ("the tails" are convergent to 0).

For $x\to-\infty$, $0\geq x\,F_X(x)=x\int_{-\infty}^{x} dF_X(t)\geq\int_{-\infty}^{x} t\,dF_X(t)\xrightarrow[x\to-\infty]{}0$, that is $\lim_{x\to-\infty} x\,F_X(x)=0$.

(Comment: The value $E[(d-X)_+]$ can be interpreted as the weight of the lower tail of the distribution of the random variable X, , i.e. the area of the domain which lies between the horizontal axes and the graph of the cumulative distribution function of X, from $-\infty$ up to d.)

One can notice that $(X-d)_+-(d-X)_+=X-d$ and so, in mean, we can write $E[(X-d)_+]=E[X]-d+E[(d-X)_+]$.

1) Let $X\in N(\mu,\sigma^2)$, $E[(X-d)_+]=\mu-d+E[(d-X)_+]=\mu-d+\int_{-\infty}^{d}F_X(x)dx=$

$$=\mu-d+\int_{-\infty}^{d}F_X(x)dx=\mu-d+\int_{-\infty}^{d}\phi\left(\frac{x-\mu}{\sigma}\right)dx=$$

$$=\mu-d+\sigma\int_{-\infty}^{\frac{d-\mu}{\sigma}}\phi(y)dy,$$ a value which can be computed using numerical methods or specialized software.

2) Let $X\in Lognorma(\mu,\sigma^2)$, $Y=\ln X, Y\in N(\mu,\sigma^2)$, or $X=e^Y$.

$$E[X]=E[e^Y]=g_Y(t)\Big|_{t=1}=e^{t\mu+\frac{t^2\sigma^2}{2}}\Big|_{t=1}=e^{\mu+\frac{\sigma^2}{2}},$$

and, for $d>0$, $E[(d-X)_+]=\int_{0}^{d}(d-x)dF_X(x)=$

$$=d\int_{0}^{d}dF_X(x)-\int_{0}^{d}x\,dF_X(x)=$$

$$=d\cdot F_X(d)-\int_{0}^{d}\frac{1}{\sigma\sqrt{2\pi}x}\,x\,e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}dx.$$

We consider the substitution $\frac{\ln x-\mu}{\sigma}=y, x=e^{\mu+\sigma y}\Rightarrow dx=\sigma e^{\mu+\sigma y}dy$,

x	0_+	d
$y=\frac{\ln x-\mu}{\sigma}$	$-\infty$	$\frac{\ln d-\mu}{\sigma}\overset{not}{=}\alpha$

$$\Rightarrow E[(d-X)_+]=d\cdot P(X\leq d)-\frac{1}{\sqrt{2\pi}}e^{\mu+\frac{1}{2}\sigma^2}\int_{-\infty}^{\alpha}e^{-\frac{1}{2}(y-\sigma)^2}dy$$

Taking $t=y-\sigma\Rightarrow y=t+\sigma, dy=dt,$

y	$-\infty$	α
$t=y-\sigma$	$-\infty$	$\alpha-\sigma$

$$\Rightarrow E[(d-X)_+]=d\cdot P(e^Y\leq d)-\frac{1}{\sqrt{2\pi}}e^{\mu+\frac{1}{2}\sigma^2}\int_{-\infty}^{\alpha-\sigma}e^{-\frac{1}{2}t^2}dt=$$

$$=d\cdot P(Y\leq\ln d)-e^{\mu+\frac{1}{2}\sigma^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha-\sigma}e^{-\frac{1}{2}t^2}dt=$$

$$= d \cdot P\left(\frac{Y-\mu}{\sigma} \le \frac{\ln d - \mu}{\sigma}\right) - e^{\mu+\frac{1}{2}\sigma^2} \cdot \phi(\alpha - \sigma) =$$

$$= d \cdot \phi(\alpha) - e^{\mu+\frac{1}{2}\sigma^2} \cdot \phi(\alpha - \sigma).$$

Finally, $E[(X-d)_+] = E[X] - d + E[(d-X)_+] =$

$$= e^{\mu+\frac{\sigma^2}{2}} - d + d \cdot \phi(\alpha) - e^{\mu+\frac{1}{2}\sigma^2} \cdot \phi(\alpha - \sigma) =$$

$$= e^{\mu+\frac{\sigma^2}{2}} \cdot (1 - \phi(\alpha - \sigma)) - d \cdot (1 - \phi(\alpha)), \text{ where } \alpha = \frac{\ln d - \mu}{\sigma}.$$

3) $(X-d)_+ = \begin{cases} X-d, & X \ge d \\ 0, & X < d \end{cases}$

If $0 < d \le \alpha \Rightarrow X \ge d \Rightarrow (X-d)_+ = X - d$

$$\Rightarrow E[(X-d)_+] = E[X-d] = \int_\alpha^\beta x \frac{1}{\beta-\alpha} dx - d = \frac{\alpha+\beta}{2} - d.$$

If $\alpha < d < \beta \Rightarrow (X-d)_+ = \max\{X-d, 0\} = \begin{cases} 0 & , \alpha < X < d \\ X-d & , d \le X < \beta \end{cases}$

$$\Rightarrow E[(X-d)_+] = \int_d^\beta (x-d) \frac{1}{\beta-\alpha} dx = \frac{(\beta-d)^2}{2(\beta-\alpha)}.$$

In the following tables we give some values of the stop-loss premiums in the case the risks follow a normal, lognormal or uniform distribution law.

1) If $X \in N(\mu, \sigma^2)$, we get

μ	σ	d	Stop-loss premium
1500	3	150(=10%* μ)	1350
1500	30	150(=10%* μ)	1350
1500	300	150(=10%* μ)	1350
1500	300	300(=20%* μ)	1200
1500000	3	150000(=10%* μ)	1350000
1500000	4000	150000(=10%* μ)	1350000
1500000	3000	375000(=25%* μ)	1125000

2) If $X \in Lognormal(\mu, \sigma^2)$, we get

μ	σ	mean	d	Stop-loss premium
7	0.1	1102.1	150	952.13
7	0.5	1242.6	150	1092.6
7	0.8	1510.2	150	1360.4

The relationship between the individual premiums and the collective premium is given in the next theorem.

Theorem The stop-loss premium corresponding to the aggregate comonotonic loss S^c for the random vector $(X_1^c, ..., X_n^c)$ is $E[(S^c - d)_+] = \sum_{i=1}^n [(X_i - d_i)_+]$, with $F_{S^c}^{-1}(0) < d < F_{S^c}^{-1}(1)$, where $d_i = F_{X_i}^{-1}(F_{S^c}(d)), i = \overline{1,n}$.

Using the above theorem, one can prove the next one.

Theorem (comonotonic upper bound for a sum of random variables)

For any random vector $(X_1, ..., X_n)$, $X_1 + X_2 + ... + X_n \le_{cx} X_1^c + X_2^c + ... + X_n^c$ holds true.

Observation: The above theorem states that the least attractive random vector $(X_1,...,X_n)$ having the marginal distribution functions $F_1 \overset{not}{=} F_{X_1}, F_2 \overset{not}{=} F_{X_2},...,F_n \overset{not}{=} F_{X_n}$ known, i.e. the sum of its components is the largest in the sense of convex order, has a joint comonotonic distribution, i.e. its common distribution is given by $\left(F_{X_1}^{-1}(U),F_{X_2}^{-1}(U),...,F_{X_n}^{-1}(U)\right)$. The components of this vector are dependent in a maximal way, because its components are expressed as functions of the same random variable.

Numerical illustration.

Let the random vector $\underline{X}=(X_1,...,X_n)$, $X_i \in Uniform(\alpha_i,\beta_i), \alpha_i<\beta_i, i=\overline{1,n}$.

We can write $X_i \overset{(R)}{=} \alpha_i+(\beta_i-\alpha_i)U, \forall i=\overline{1,n}$

$\Rightarrow$ there exist the random variable $U \in Uniform(0,1)$

and there exist the strictly increasing functions $f_i(u)=\alpha_i+(\beta_i-\alpha_i)u, i=\overline{1,n}$.

$$F_{X_i}(x)=\begin{cases}0 & ,x\le\alpha_i\\ \dfrac{x-\alpha_i}{\beta_i-\alpha_i} & ,\alpha_i<x<\beta_i\\ 1 & ,x\ge\beta_i\end{cases}$$

and $F_{X_i}^{-1}(p)=\alpha_i+(\beta_i-\alpha_i)p,\ p\in(0,1), i=\overline{1,n}$.

$$\Rightarrow S^c=X_1^c+...+X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U)+...+F_{X_n}^{-1}(U)=$$

$$=\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}(\beta_i-\alpha_i)U,$$

thus $S^c \in Uniform\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\beta_i\right)$ and $S\le_{cx} S^c$.

Let us consider a portfolio of non-independent $n=100$ risks $\underline{X}=(X_1,...,X_n)$ such that: $X_1,...,X_{n_1}$, with $n_1=35$ are identical distributed with $Y_1 \in Uniform(\alpha_1,\beta_1)$, $(\alpha_1,\beta_1)=(10,23)$, $X_{n_1+1},...,X_{n_1+n_2}$, with $n_2=45$, are identical distributed with $Y_2 \in Uniform(\alpha_2,\beta_2)$, $(\alpha_2,\beta_2)=(8,17)$, and $X_{n_1+n_2+1},...,X_n$, with $n_3=20$, are identical distributed with $Y_3 \in Uniform(\alpha_3,\beta_3)$, $(\alpha_3,\beta_3)=(13,25)$,

The comonotonic sum

$$S^c=X_1^c+...+X_n^c \overset{(R)}{=} F_{X_1}^{-1}(U)+...+F_{X_n}^{-1}(U)=n_1Y_1+n_2Y_2+n_3Y_3=$$

$$=\sum_{k=1}^{3}n_k\alpha_k+\sum_{k=1}^{3}n_k(\beta_k-\alpha_k)U$$

$\Rightarrow S^c \in Uniform\left(\sum_{k=1}^{3}n_k\alpha_k,\sum_{k=1}^{3}n_k\beta_k\right)$. i.e. $S^c \in Uniform(\alpha=970,\beta=2070)$,

and the aggregate loss $X_1+X_2+...+X_n \le_{cx} S^c$, so the stop-loss premiums verify the inequality $E\left[(X_1+X_2+...+X_n-d)_+\right]\le E\left[(S^c-d)_+\right]$, for any deductible d.

Let $d=1200\in(970,2070)$, then the stop-loss premium for the aggregate conomotonic loss of this portfolio is $E\left[(S^c-d)_+\right]=\dfrac{(\beta-d)^2}{2(\beta-\alpha)}=344.0454545$.

$E\left[\left(S^c-d\right)_+\right]=\sum_{i=1}^{n}E\left[\left(X_i-d_i\right)_+\right]=\sum_{k=1}^{3}n_kE\left[\left(Y_k-d_k\right)_+\right]$, where the deductibles are found from $d_k=F_{Y_k}^{-1}\left(F_{S^c}\left(d\right)\right), k=1,2,3.$

For $d\in(\alpha,\beta), F_{S^c}(d)=\dfrac{d-\alpha}{\beta-\alpha}\in(0,1)$ and $F_{Y_k}^{-1}(p)=\alpha_k+(\beta_k-\alpha_k)p,\ p\in(0,1), k=1,2,3$

$$\Rightarrow d_k=\alpha_k+(\beta_k-\alpha_k)\frac{d-\alpha}{\beta-\alpha},\ k=1,2,3,$$

So $d_1=12.71818182$, $d_2=9.881818182$, $d_3=15.50909091$ and one may check that $n_1d_1+n_2d_2+n_3d_3=1200=d$.

Finally, we can compute the individual stop-loss premiums.

For $k=\overline{1,35}, E\left[\left(X_k-d_1\right)_+\right]=E\left[\left(Y_1-d_1\right)_+\right]=\dfrac{(\beta_1-d_1)^2}{2(\beta_1-\alpha_1)}=4.065991734$,

for $j=\overline{36,80}, E\left[\left(X_j-d_2\right)_+\right]=E\left[\left(Y_2-d_2\right)_+\right]=\dfrac{(\beta_2-d_2)^2}{2(\beta_2-\alpha_2)}=2.814917355$,

and for $l=\overline{81,100}, E\left[\left(X_l-d_3\right)_+\right]=E\left[\left(Y_3-d_3\right)_+\right]=\dfrac{(\beta_3-d_3)^2}{2(\beta_3-\alpha_3)}=3.75322314$.

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Analytical Revision on the Proofs for Comonotone Additvity and Sub-additivity of Distorted Risk Measures

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Abstract.
In financial and insurance markets no-arbitrage argument is an important condition which can be achieved by additivity property in suggested risk measures and pricing models. In this paper, I have provided some discussions about revision of previous proofs for addtitivity of dependent comonotone risks and sub-additivity property of premium principles under distortion. Four defined properties of a distortion operator in hand, I have bring a complete proof for additivity of comonotone risks in distorted risk measures which may be used as a premium principle in insurance. The key concept in the proof is that $S_{T(x)}^{-1}(p) = T(S_X^{-1}(p))$, where $T: R \to R$ is an increasing continuous function and S^{-1} is generalized inverse function of decumulative distribution function. I examined also the provided proof of sub-additivity by Wirch and Hardy, 1999 and complete the relative theorems.

Keywords: additivity, sub-additivity, distortion operator, premium principle, decumulative distribution function, correlation order, stop-loss order.

Introduction

By a simple definition, a risk measure is a function that allocates a non-negative real number to a risk. Many risk measures have been suggested to quantifying financial and insurance risks, but there are some important considerations to measure the insurance risks which are not the alike with the financial risk measuring. Financial pricing models cannot be applied truly for pricing insurance risks, because of some fundamental differences between these two types.

Distorted risk measured have been introduced and developed in order to find a universal framework for pricing financial and insurance risks. Considerable efforts have been made by actuaries and financial economists to build bridges to connect financial and insurance pricing theories, which have been developed after Yaari's dual theory under risk in 1987. see D'Arcy and Doherty (1988), Gerber and Shiu (1994). put together various pieces of pricing theory puzzles, an overall picture has not emerged yet.

In the insurance literature, the appropriate risk measures to pricing insurance risks are premium principles. By an (insurance) risk we mean a non-negative random variable representing the amount paid to a policyholder or his/her beneficiary, and/or a third party. We denote the set of such random variables by Υ. The premium related to an insurance risk (i.e. loss random variable) $X \in \Upsilon$ will denoted by $H(X)$.

There exist some mathematical and economical requirements such as extra loading, maximal loss, translation and scale invariance, monotonicity, first-order stochastic dominance, stop-loss ordering and continuity for premium principles which are needed to meet natural conditions of insurance contracts and ratemaking approaches while some others are just desirable.

Traditional premium principles such as Statndard deviation, Exponential, Dutch, Swiss principles and principle of Equivalent Utility couldn't preserve some of basic requirements of a desirable premium principle especially extra loading and commonly were based on the first two moments. Since loss distributions are often skewed, first two moments are not completely efficient about the level of the insurance risk. For confronting and

dealing with above problems and inefficiencies in referred principles, Yaari (1987) postulated that decision makers should adjust their assumed probability distributions according to their risk-aversion for probabilities, and then make decision based on that adjusted probability distribution.

This idea leads to using a general class of "**Distortion Operators**" defined by function g with some particular properties for pricing both financial and insurance risks. Considerable efforts have been made by actuaries and financial economists to build bridges to connect financial and insurance pricing theories, which have been developed after Yaari's dual theory under risk in 1987. see D'Arcy and Doherty (1988), Gerber and Shiu (1994).

In and special view, Venter's (1991) observed that such an integration between insurance and financial pricing models may have come into reality by transforming the distribution of risk and proved it in case of insurance layers. Wang (1995, 1996) proposed calculating insurance premium by transforming the decumulative distribution function, which turned out to coincide with Yaari's economic theory of risk. So, the related premium principle have been defined by,

$$H_g[X] = -\int_{-\infty}^{0}\{1 - g[S_X(t)]\}dt + \int_{0}^{\infty} g[S_X(t)]dt$$

where for a non-negative random variable (insurance risk) it turns into

$$H_g[X] = \int_{0}^{\infty} g[S_X(t)]dt$$

The distorted risk measure in insurance should satisfy the extra loading necessity;

$$H_g[X] \leq E[X]$$

But, the vital consideration for new pricing framework under distorted risk measure was preserving the No-arbitrage argument for calculated prices in the market. No-arbitrage argument has direct relationship with additivity property of premium principle as a risk measures. In addition, the sub-additivity property is too notable in premium evaluation and coherency or proposed risk measure.

Many actuaries have tried to verify the additivity property in general for dependent and independent risks. Although this attempts was failed for dependent risks, but they could prove additivity for a large array of risk called **Comonotone Risks**. Wang (1995) and Kaas, Dhaene, Vyncke, Goovaerts, Denuit have done many researches on this subject during 1995 to 2004 which had been laid those foundation by a proposition by Denneberg (1994). On the other hand, for the case of sub-additivity there are some proofs by Wang (1995), Wirch & Hardy (2000) for arbitrary discrete points risk random variable. Finally, Dhaene, Vanduffel, Goovaerts, kaas & Vyncke proposed an alternative theorem for proving sub-additivity of distorted risk measures by using the comonotone additivity property and stop-loss ordering.

In this paper we are going to rearrange the complete proofs for both additivity and sub-additivity or distorted risk measures specialized in premium principles with a brief examination of previous proofs.

Distortion Operator

Definition

A function $g : [0,1] \rightarrow [0,1]$ is called a distortion function if it satisfies the following four conditions:

1. $0 \leq g(u) \leq 1$ and that $g(0) = 0$ and $g(1) = 1$.
2. g is increasing (so $g'(u) \geq 0$ when the derivative exists).
3. g is concave (so $g''(u) \leq 0$ when the second derivative exists).
4. $g'(0) = +\infty$

The resulting premium would be risk-adjusted if the value of $S_Y(t) = g[S_X(t)]$ dominates $S_X(t)$ sufficiently. The transformed decumulative function $S_Y(t)$ consists of the extra risk premium or the risk loading that adjusts the net premium obtained from the distribution of original risk X (i.e. $S_X(t)$).

Generally the distortion operator $g : [0,1] \rightarrow [0,1]$ transforms a probability distribution S_X to a new distribution S_Y . As we should be able to transform the original risk distribution, we should also be able to transform the layers' distributions at the same time, because in insurance applications, the insurers often have to determine premiums for layers. For insurance layer $X(a, a+h]$ the risk-adjusted premium easily can be obtained from:

$$H_g[X(a, a + h]] = \int_0^{\infty} g[S_{X(a,a+h]}(t)]dt = \int_a^{a+h} g[S_X(t)]dt$$

indicating that the layer premium can be obtained by applying distortion operator g either to the distribution of the layer or to the distribution of the original risk X (ground-up loss).

These assumed properties are necessary to construct a desirable premium principle. Based on some principles of insurance pricing, premium principles should meet some requirements: For example, it is desirable that for projected high loss values a premium principle should carry more risk loading than low loss values. The following observations come along with the above conditions the distortion function has:

Note I. It is known that a concave function $f : (a, b) \to \mathbb{R}$ is necessarily continuous. Thus, function g is continuous on $(0,1)$.

Note II. As g being increasing and $S_X(x)$ being decreasing, $g[S_X(x)]$ is a decreasing function.

Note III.

$$\lim_{x\to+\infty} g[S_X(x)] = g[\lim_{x\to+\infty} S_X(x)] = g(0) = 0$$
$$\lim_{x\to-\infty} g[S_X(x)] = g[\lim_{x\to-\infty} S_X(x)] = g(1) = 1$$

Note that $g[S_X(x)]$ is not necessarily a decumulative function unless we have right-continuity condition of g.

Note IV. g being continuous (Note I) and $S_X(x)$ being right-continuous, it implies that $g[S_X(x)]$ is right continuous on $(0,1)$.

Note V. It is notable that $g[S_X(x)]$ is right continuous everywhere if we assume g being continuous at 1.To see this, suppose $x \to x_0^+$; we must show that $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$. For this, distinguish three cases:

Case I. $S_X(x_0)] = 0$. Then, for all $x > x_0$, we have $S_X(x)] = 0$ as S_X is a decreasing function. Then $g[S_X(x)] = g[S_X(x_0)]$ for $(x > x_0)$ giving $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$.

Case II. $0 < S_X(x_0) < 1$. then from the right continuity of S_X, we have $0 < S_X(x) < 1$ for x's on the right side of x_0 and close to x_0. Further, we have $\lim_{x\to x_0^+} S_X(x) = S_X(x_0)$. Now the continuity of g on $(0,1)$ gives $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$.

Case III. $S_X(x_0)] = 1$. Then, from the right continuity of S_X and the continuity of g at 1, one gets $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$, finishing the claim in all cases.

Additivity of Comonotone Risks

Although $H_g[X]$ is not additive in general and may not meet the no-arbitrage argument in pricing framework under $H_g[X]$, but it preserves additivity for comonotone risks. In other word, distorted premium principle is additive for dependent comonotone risks. Consequently, for insurance layers, the resulted premium will be additive.

Definition (Comonotone Risk)

Two random variables X1 and X2 are comonotone risks if there exist a risk Z and weakly increasing functions f and g such that $X_1 = f(Z)$ and $X_2 = g(Z)$.

Comonotone risks cannot be hedged against each other. In the no-hedged condition, insurers are not willing to give a reduction in the risk load for a combined policy of two comonotone risks X_1 and X_2, thus $H(X_1 + X_2) \geq H(X_1) + H(X_2)$. On the other hand the maximum premium that insurers can charge for such a combined policy is the sum of two individual risk premiums, because otherwise the policy-holder can buy separate covers cheaper; thus we must have $H(X_1 + X_2) \geq H(X_1) + H(X_2)$. These two statements imply additivity for the comonotone risks:

$$H(X_1 + X_2) = H(X_1) + H(X_2)$$

Definition

Let, $X:\ \Omega \to \overline{\mathbb{R}}$ be a function on the set Ω. By the upper set system of X we mean the collection

$$\mathcal{M} = \{\{X > x\}\big|\ x \in \overline{\mathbb{R}}\} \cup \{\{X \geq x\}|x \in \overline{\mathbb{R}}\}$$

An analogous to definition 0 is as below

Definition

A class of $\mathcal{L}$ of functions $\Omega \to \overline{\mathbb{R}}$ is called comonotone if the union $\cup_{X\in\mathcal{L}}\ \mathcal{M}_X$ is a chain.

Clearly, a class of functions is comonotone, if and only if, each pair of functions in $\mathcal{L}$ is comonotone. The following proposition gives equivalent conditions for a pair of functions to be comonotone.

Proposition

For two functions $X, Y:\ \Omega \to \overline{\mathbb{R}}$ the following conditions are equivalent:

(i) $\{X, Y\}$ is comonotone.

(ii) There is no pair $\omega_1, \omega_2 \in \Omega$ such that

$$\begin{cases} X(\omega_1) < X(\omega_2) \\ \\ Y(\omega_1) < Y(\omega_2) \end{cases}$$

(iii) The set $\{(X(\omega), Y(\omega))\big|\omega \in \Omega\} \subset \overline{\mathbb{R}}^2$ is a chain with respect to the usual relation in $\overline{\mathbb{R}}^2$.

(iv) There exist a function $Z:\ \Omega \to \overline{\mathbb{R}}$ and increasing functions u and v on $\overline{\mathbb{R}}$ such that

$$X = u(Z) \quad , \quad Y = v(Z)$$

(v) There exist continuous and increasing functions u and v on $\overline{\mathbb{R}}$ such that $u(Z) + v(Z) = Z, \quad (Z \in \overline{\mathbb{R}})$ and

$$X = u(X + Y) \quad , \quad Y = v(X + Y)$$

Proof:

(i) ⇒ (ii)

Suppose **(ii)** is not true. Then there are $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) < X(\omega_2)$ and $Y(\omega_1) < Y(\omega_2)$. Set $A = \{X > X(\omega_1)\}$ and $B = \{Y > Y(\omega_2)\}$. Then, $\omega_2 \in \mathrm{A}\backslash\mathrm{B}$ and $\omega_1 \in \mathrm{B}\backslash\mathrm{A}$, while A and B are in $\mathcal{M}_X \cup \mathcal{M}_Y$. This shows that $\mathcal{M}_X \cup \mathcal{M}_Y$ is not a chain rejecting **(i)**.

(iii) ⇒ (i)

Assume on the contrary **(i)** is not true. Then there are $A \subset \mathcal{M}_X$ and $B \subset \mathcal{M}_Y$ satisfying $A \not\subseteq B$ and $B \not\subseteq A$. Choose , $\omega_1 \in \mathrm{A}\backslash\mathrm{B}$ and $\omega_2 \in \mathrm{B}\backslash\mathrm{A}$. Either $A = \{X|X > a\}$ or $A = \{X|X \geq a$. In the first case, $\omega_1 \in \mathrm{A}$ and $\omega_2 \notin \mathrm{A}$ we get $\begin{cases} X(\omega_1) > a, \\ \\ X(\omega_2) \leq a \end{cases}$ giving us the inequality $X(\omega_1) > X(\omega_2)$. In the second case, from $\omega_1 \in \mathrm{A}$ and $\omega_2 \notin \mathrm{A}$ we have $\begin{cases} X(\omega_1) \geq a, \\ \\ X(\omega_2) < a \end{cases}$ giving us the inequality $X(\omega_1) > X(\omega_2)$. In either cases, we have $X(\omega_1) > X(\omega_2)$. Similarly, we have $Y(\omega_1) > Y(\omega_2)$. Those two reject **(ii)**.

(ii) ⇒ (v)

Assume there are ω_1 and ω_2 such that

$$X(\omega_1) + Y(\omega_1) = X(\omega_2) + Y(\omega_2)$$
$$X(\omega_1) - X(\omega_2) = Y(\omega_1) - Y(\omega_2)$$

If we had $X(\omega_1) < X(\omega_2)$, then we would have $Y(\omega_2) < Y(\omega_1)$ contradicting **(ii)**. So, $X(\omega_1) \geq X(\omega_2)$. Similarly, the case $X(\omega_1) > X(\omega_2)$ ends up in a contradiction. Therefore,

$$X(\omega_1) = X(\omega_2)$$

and

$$Y(\omega_1) = Y(\omega_2)$$

Now set $Z = X + Y$. Then the functions $\begin{cases} u: Z(\Omega) \longrightarrow \mathbb{R}, \\ \\ v: Z(\Omega) \longrightarrow \mathbb{R} \end{cases}$ defined by $\begin{cases} u(Z(\omega)) = X(\omega), \\ \\ v(Z(\omega)) = Y(\omega) \end{cases}$ are well defined. Next we prove u and v are increasing;

Suppose $Z(\omega_1) < Z(\omega_2)$. Then

$$X(\omega_1) - X(\omega_2) < Y(\omega_2) - Y(\omega_1)$$

If the right-hand side were negative, then the left-hand side is negative too, contradicting **(ii)**. Similarly, the left-hand side cannot be positive. Thus, we must have $\begin{cases} X(\omega_1) - X(\omega_2) \leq 0, \\ \\ Y(\omega_2) - Y(\omega_1) \geq 0 \end{cases}$ i.e. $\begin{cases} X(\omega_1) \leq X(\omega_2), \\ \\ Y(\omega_1) \leq Y(\omega_2) \end{cases}$

so $\begin{cases} u(Z(\omega_1)) \leq u(Z(\omega_2)), \\ \\ v(Z(\omega_1)) \leq v(Z(\omega_2)) \end{cases}$.

This proves u and v being increasing. Next, we show that u and v are continuous on $Z(\Omega)$. For this take two elements $Z(\omega_1)$ and $Z(\omega_2)$ from $Z(\Omega)$ where $Z(\omega_1) \leq Z(\omega_2)$. As we sow above,

$$\begin{cases} X(\omega_1) \leq X(\omega_2), \\ \\ Y(\omega_1) \leq Y(\omega_2) \end{cases}$$

so,

$$0 \leq X(\omega_2) - X(\omega_1) = Z(\omega_2) - Z(\omega_1) - \underbrace{(Y(\omega_2) - Y(\omega_1))}$$
$$\leq Z(\omega_2) - Z(\omega_1)$$

Where under braced value is always positive. Thus,

$$0 \leq X(\omega_2) - X(\omega_1) = [Z(\omega_2) - Z(\omega_1)]$$
$$0 \leq u(Z(\omega_2)) - u(Z(\omega_1)) = [Z(\omega_2) - Z(\omega_1)]$$

This shows that if $|Z(\omega_2) - Z(\omega_1)| \to 0$ then $|u(Z(\omega_2)) - u(Z(\omega_1))| \to 0$. So, u is uniformly continuous on $Z(\Omega)$. This allows extending u to a continuous function on $\overline{Z(\Omega)}$. Now the complement $\mathbb{R} \backslash \overline{Z(\Omega)}$ is a countable union of intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$. On each (a_n, b_n) define u linearly maintaining $u(a_n)$ and $u(b_n)$. Then define v on (a_n, b_n) by $v(Z) = Z - u(Z)$. Then, u and v are continuous on $\mathbb{R}$ satisfying $u(Z) + v(Z) = Z$. With this extensions, we have $X = u(X + Y)$ and $Y = v(X + Y)$ on Ω.

(v) ⇒ (iv)

This is obvious.

(iv) ⇒ (iii)

If $\omega_1, \omega_2 \in \Omega$, then either $Z(\omega_1) \leq Z(\omega_2)$ or $Z(\omega_2) \leq Z(\omega_1)$. In the first case, we have $\begin{cases} u(Z(\omega_1)) \leq u(Z(\omega_2)), \\ \\ v(Z(\omega_1)) \leq v(Z(\omega_2)) \end{cases}$ i.e. $\begin{cases} X(\omega_1) \leq X(\omega_2), \\ \\ Y(\omega_1) \leq Y(\omega_2) \end{cases}$, while in the second case we have the opposite direction. This proves **(iii)**. This finishes the proof of the proposition.

■

Definition

For a decumulative distribution function S we define the **Generalized Inverse Function** as being

$$S^{-1}(p) = sup\{x | S(x) > p\} \qquad (0 < p < 1)$$

Lemma

$$S^{-1}(p) \leq x \quad \Leftrightarrow \quad S(x) \leq p$$

Proof:

If $x^* < S^{-1}(p)$, then from the definition of sup, the exists some x such that $S(x) > p$ and $x^* < x$. Then $S(x^*) \geq S(x) > p$ as S is decreasing. So $S(x^*) > p$. This proves one half of the claim. For the other half, suppose $S(x) > p$. Then from the right continuity of S, we have $S(x+\varepsilon) > p$ for some $\varepsilon > 0$. So $x + \varepsilon$ is in the set $\{x \mid S(x) > p\}$. Therefore, it must be bounded by the Sup of the set, that is $x + \varepsilon \leq S^{-1}(p)$. Therefore, $x < S^{-1}(p)$. This proves the other half.

■

This lemma shows that

$$S^{-1}(p) = \inf\{x \mid S^{-1}(p) \leq x\} = \inf\{x \mid S(x) \leq p\}$$

So,

$$S^{-1}(p) = \inf\{x \mid S(x) \leq p\} = \sup\{x \mid S(x) > p\}$$

For any increasing right-continuous function T, we define the generalized inverse T^{-1} by

$$T^{-1}(q) = \inf\{x \mid T(x) \geq q\}$$

The inf is taken to be $+\infty$ should the set be empty. Parts (i) and (ii) of the following proposition can be prove in a similar manner as above. So we only prove part (iii):

Proposition

For an increasing right-continuous function $T: \mathbb{R} \to \mathbb{R}$,

(i)

$$T^{-1}(q) = inf\{x \mid T(x) \geq q\} = sup\{x \mid T(x) < q\}$$

(ii)

$$T^{-1}(q) \leq x \quad \Leftrightarrow \quad q \leq T(x)$$

(iii) T^{-1} is increasing and left-continuous.

Proof:

(iii)

If $p < q$, then by letting $A_p = \{x \mid T(x) \geq p\}$ and $A_q = \{x \mid T(x) \geq q\}$, we have

$$A_p \supset A_q$$
$$inf\, A_p \leq inf\, A_q$$
$$T^{-1}(p) \leq T^{-1}(q)$$

So that T^{-1} is an increasing function. To see left-continuity, let $p_n \to p^-$. Since $p_n < p$ and since T^{-1} in increasing, for all n we have $T^{-1}(p_n) \leq T^{-1}(p)$. Therefore,

$$\boldsymbol{lim\, T^{-1}(p_n) \leq T^{-1}(p)} \qquad \textbf{(*)}$$

Now let γ be any number satisfying $\gamma > T^{-1}(p)$. Then from the definition of inf, there exists some x such that

$$\begin{cases} p \leq T(x), \\ x < \gamma \end{cases}$$

But from $T(x) \geq p$ we equivalently have $T^{-1}(p) \leq x$. So we can further write

$$\begin{cases} \boldsymbol{p \leq T(x)}, \\ \boldsymbol{T^{-1}(p) \leq x < \gamma} \end{cases} \qquad \textbf{(**)}$$

Now we have the following series of conclusions:

$$x-\gamma+T^{-1}(p)<T^{-1}(p)$$
$$T\left(x-\gamma+T^{-1}(p)\right)<p$$

$$T\left(x-\gamma+T^{-1}(p)\right)<p_n \qquad \text{for large n}$$

$$x-\gamma+T^{-1}(p)<T^{-1}(p_n) \quad \text{for large n}$$

So,

$$\boldsymbol{x-\gamma+T^{-1}(p)<\underline{\lim}\ T^{-1}(p_n)} \qquad \textbf{(***)}$$

Now by letting $\gamma \to T^{-1}(p)$, we have $x \to T^{-1}(p)$ in (**), therefore (***) reduces to

$$\boldsymbol{T^{-1}(p)<\underline{\lim}\ T^{-1}(p_n)} \qquad \textbf{(****)}$$

Now (*) and (****) imply $limT^{-1}(p_n)=T^{-1}(p)$ when $p_n \to p$, finishing the proof of part (iii).

Lemma

If T is continuous, then T^{-1} is one-to-one (and so it is strictly increasing).

Proof:

Suppose $p<q$. We want to show that $T^{-1}(p)<T^{-1}(q)$. First of all, we know that T^{-1} is increasing: $T^{-1}(p)\leq T^{-1}(q)$. So if $T^{-1}(p)<T^{-1}(q)$ does not hold, then we must have had $T^{-1}(p)=T^{-1}(q)$. Call this common value x. Then from $T^{-1}(p)<x$, we have $p\leq T(x)$. On the other hand, from the continuity of T^{-1},

$$x=T^{-1}(p)$$
$$x-\frac{1}{n}<T^{-1}(p) \qquad \forall n$$
$$T\left(x-\frac{1}{n}\right)<p \qquad \forall n$$
$$T(x)\leq p \qquad \text{(continuity of T)}$$

The two inequalities show that $T(x)=p$. Similarly, one gets $T(x)=q$. So, $p=q$, a contradiction.

■

Proposition

Suppose $T:\ \mathbb{R}\to\mathbb{R}$ is an increasing continuous function. Then

$$S^{-1}_{T(X)}(p)=T(S^{-1}_X(p))$$

Proof:

It suffices to show that for every x the relation $S^{-1}_{T(X)}(p)\leq x$ holds if and only if the relation $T(S^{-1}_X(p))\leq x$ holds. Further, this remains true if we prove it for all x but a countable number of x's. Since T^{-1} is an increasing function, its set of discontinuities is countable. So we may attempt proving this assertion for the continuity points x of T^{-1}: Now with this assumption in mind, for every $o<p<1$, we then have the following equivalent expressions:

$$
\begin{aligned}
S_{T(X)}^{-1}(p) \le x \quad &\Leftrightarrow \quad S_{T(X)}(x) \le p \\
&\Leftrightarrow \quad \Pr[T(X) > x] \le p \\
&\Leftrightarrow \quad 1 - p \le \Pr[T(X) \le x] \\
&\Leftrightarrow \quad 1 - p \le \lim_{n \to \infty} \Pr\left[T\left(X - \frac{1}{n}\right) < x\right] \qquad [\text{as } T \text{ is continuous}] \\
&\Leftrightarrow \quad 1 - p \le \lim_{n \to \infty} \Pr\left[X - \frac{1}{n} < T^{-1}(x)\right] \\
&\Leftrightarrow \quad 1 - p \le \Pr[X \le T^{-1}(x)] \\
&\Leftrightarrow \quad 1 - p \le 1 - \Pr[X > T^{-1}(x)] \\
&\Leftrightarrow \quad S_X(T^{-1}(x)) \le p \\
&\Leftrightarrow \quad S_X^{-1}(p) \le T^{-1}(x) \\
&\Leftrightarrow \quad \forall n: \quad S_X^{-1}(p) < T^{-1}\left(x + \frac{1}{n}\right) \\
&\qquad \text{as } T^{-1} \text{ is both strictly increasing and continuous at } x \\
&\Leftrightarrow \quad \forall n: \quad T(S_X^{-1}(p)) < \left(x + \frac{1}{n}\right) \\
&\Leftrightarrow \quad T(S_X^{-1}(p)) \le x
\end{aligned}
$$

■

Theorem

If $\{X, Y\}$ is comonotone, then for $Z = X + Y$, we have

$$S_Z^{-1}(p) = S_X^{-1}(p) + S_Y^{-1}(p).$$

Proof:

As in the above theorem, there are continuous function u and v on $\mathbb{R}$ such that $u + v$ is the identity function and that

$$\begin{cases} X = u(X+Y), \\ Y = v(X+Y) \end{cases}.$$

Then

$$
\begin{aligned}
S_X^{-1}(p) + S_Y^{-1}(p) &= S_{u(X+Y)}^{-1}(p) + S_{v(X+Y)}^{-1}(p) \\
&= u[S_{X+Y}^{-1}(p)] + v[S_{X+Y}^{-1}(p)] \\
&= S_{X+Y}^{-1}(p)
\end{aligned}
$$

■

This result easily extends to the multivariate case: if $X_1, X_2, \ldots, X_{n-1}, X_n$ are mutually comonotones, then

$$S_{(X_1, X_2, \ldots, X_{n-1}, X_n)}^{-1}(p) = \sum_{i=1}^{n} [S_{X_i}^{-1}(p)]$$

Recall that the net expected loss $E(X)$ is equal to the area below the curve S_X(t). While vertical slicing gives, $E(X) = \int_0^\infty S_X(t)dt$. On the other hand, horizontal slicing yields that,

$$E(X) = \int_0^\infty S_X^{-1}(q)dq$$

where the reverse $S_X^{-1}(q)$ my not be uniquely defined. However, it does not affect the integration. See figure (0.1).

Remark. One can easily verify that for an increasing function g with $g(0) = 0$ and $g(1) = 1$,

$$H[X] = \int_0^\infty g[S_X(t)]dt = \int_0^\infty S_X^{-1}(q)dg(q)$$

with the aid of horizontal slicing, we can now examine the concept of comonotoneity in terms of insurance risk premium densities.

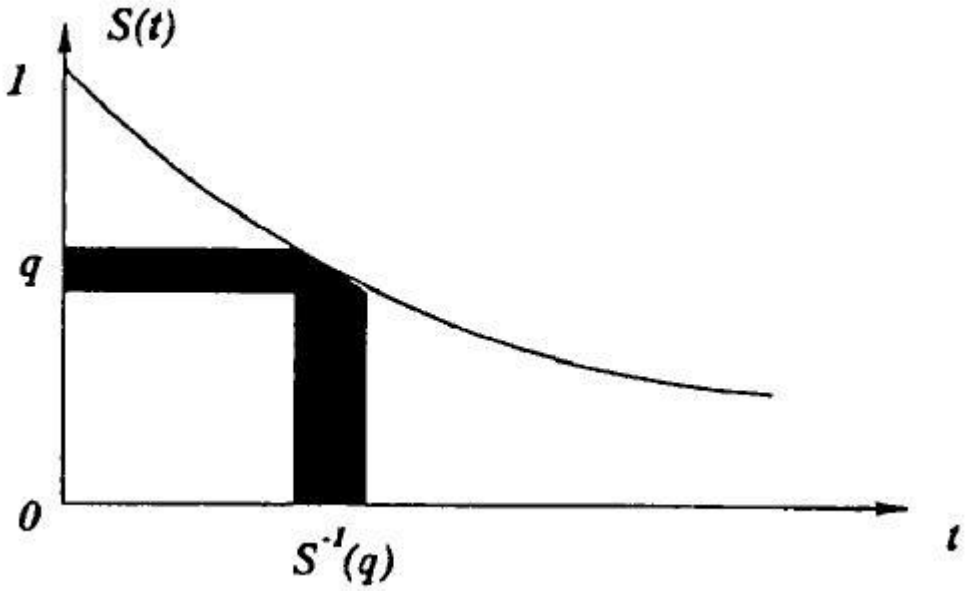


Figure 0.1: Slicing a risk vertically & horizontally

Theorem

The premium principle $H_g[X]$ is additive on comonotone risks.

Proof:

$$H(X+Y) = \int_0^\infty S_{X+Y}^{-1}(t)dg(t) = \int_0^\infty [S_X^{-1}(t) + S_Y^{-1}(t)]dg(t) = H(X) + H(Y)$$

■

Sub-additivity

Sub-additivity means the premium for the sum of two risks is not greater than the sum of the individual premiums and the rationale behind sub-additivity can be summarized as "a merger does not create extra risk". The property is desirable, because the buyer of insurance will not buy two merged insurance cover for two risk in a price more than two separate insurance cover from two insurer.

By general attitude joining financial literature, sub-additivity reflects the idea that risk can be reduced by diversification. When equality holds, two risks are additive. The diversification effect in its simplest form is defined as the difference between the sum of two premium (generally risk measures) of standalone risks and the premium of both risks taken together, that is,

$$H(X) + H(Y) - H(X+Y)$$

The diversification effect is always positive for subadditive risk measures. In portfolio optimization, sub-additivity and scale invariance ensures that the risk surface to be minimized in the space of portfolio is convex. Only if the surface is convex there would be a unique absolute minimum and only then will the risk minimization process always find a unique, well-diversified, optimal solution. In other word, validity of sub-additivity leaves no room for the policy-holder to split a risk into pieces.

We turn to discuss some of previous attempts to prove the sub-additivity of distorted risk measures.

Discussions of sub-additivity

Wang (1995) presented a theorem for sub-additivity of a distorted risk measure where the distortion function was defined especially as $g[S_X(x)] = [S_X(x)]^{1/p}$ where $p > 1$.

Theorem (Wang 1995)

For any two non-negative random variables U and V without assuming independence, the following inequality holds:

$$H_g(U+V) \leq H_g(U) + H_g(V)$$

The proof was based on selecting some arbitrary points U and V from a discrete domain of risk random variable $\{0,1,\dots,n\}$ and proving the theorem by induction for these points. Then Wang have generalized the proof for all values by the approximation which can be achieved by scale and translation invariance property. By the way, he generalized the sub-additivity to an extended domain of U and V as $U, V \in \{kh, (k+1)h, \dots, (n+k)h\}$ where $h > 0$ and $k \in \mathbb{N}$.

It's clear that the proof is not completely analytical and generalization by approximation is not sufficient.

On the other hand, Wirch & Hardy (2000) suggested another theorem for sub-additivity of distorted risk measures by using concavity of distortion function.

Theorem (Wirch & Hardy, 2000)

The distortion risk measure $Hg(X) = \int_0^\infty g(SX(x))dx$ is sub-additive if and only if, g is a concave distortion function.

In the proof, authors brought some arguments in continuation of Wang (1995) and Wang (1996), which all are based on a theorem in Denneberg (1994) and is repeated in Kaas et al (2005). There have been chosen some amounts of g and then it has been proved that for a convex function of g there would be no place for sub-additivity for these chosen values. This completes the one way of theorem (if) and although, the referred terms are strictly true, but the proof is incomplete and vital point is that there is no proof for "only if" term!

By the way, it is necessary to bring a complete proof of sub-additivity property of distorted risk measures that is specialized here for distorted premium principle H_g.

We now turn to show that the premium principle H_g is subadditive. The proof is closely base on the theorems and propositions in Wang & Dhaene (1998) and (Dhaene, et al; 1999).

Lemma

Suppose that X is a risk random variable and that U is a random variable which is uniformly distributed over the interval $[0,1]$. Then X and $F_X^{-1}(U)$ have the same distribution.

Proof:

For this we must show that for every x,

$$\Pr(F_X^{-1}(U) \leq x) = \Pr(X \leq x)$$

but from the definition of the generalized inverse we have

$$\begin{aligned} \Pr(F_X^{-1}(U) \leq x) &= \Pr(U \leq F_X(x)) \\ &= F_X(x) \\ &= \Pr(X \leq x) \end{aligned}$$

proving the claim.

■

Lemma

For every two risk variables X and Y and for U which is uniformly distributed over $[0,1]$, the variables $\{F_X^{-1}(U), F_Y^{-1}(U)\}$ are comonotone.

Proof:

In fact, the functions F_X^{-1} and F_Y^{-1} are increasing functions. Therefore, by a theorem above the random variables $F_X^{-1}(U)$ and $F_Y^{-1}(U)$ are comonotone.

■

Know consider the below proposition:

Proposition

For any two random variables X and Y and for the uniformly distributed $U \sim Uniform(0,1)$, we have the following inequality between the joint distributions

$$F_{X,Y}(x,y) \leq F_{F_X^{-1}(U),F_Y^{-1}(U)}(x,y)$$

Proof:

$$\begin{aligned} F_{X,Y}(x,y) &= \Pr(X \leq x, Y \leq y) \\ &\leq \begin{cases} \Pr(Y \leq y) \\ \Pr(Y \leq y) \end{cases} \end{aligned}$$

So,

$$F_{X,Y}(x,y) \leq \min\{F_X(x), F_Y(y)\}$$

Now,

$$\begin{aligned} F_{F_X^{-1}(U),F_Y^{-1}(U)}(x,y) &= \Pr[F_X^{-1}(U) \leq x,\ F_Y^{-1}(U) \leq y] \\ &= \Pr[U \leq F_X(x), U \leq F_Y(y)] \\ &= \Pr[U \leq \min\{F_X(x), F_Y(y)\}] \\ &= \min\{F_X(x), F_Y(y)\} \qquad (U \text{ is Uniformly distributed}) \\ &\geq F_{X,Y}(x,y) \end{aligned}$$

■

Remark: In the literature, this fact is deviated by **Correlation Order** as,

$$(X,Y) \leq_{Corr} (F_X^{-1}(U), F_Y^{-1}(U))$$

0-1

Lemma

For $\forall d > 0$ and any two random variables X and Y we have

$$E[(X+Y-d)^+] = E(X) + E(Y) - d + \int_0^d F_{X,Y}(t, d-t)dt$$

Proof:

If $\begin{cases} 0 \leq d-x-y, \\ 0 \leq x, 0 \leq y \end{cases}$, then $x \leq d-y$. So then,

$$\int_0^d I_{[x,d-y]}(t)dt = \int_x^{d-y} dt = d-x-y = (d-x-y)^+.$$

But if $\begin{cases} 0 \leq d-x-y, \\ 0 \leq x, 0 \leq y \end{cases}$, then the interval $[x, d-y]$ is empty, so $I_{[x,d-y]} = 0$. Thus,

$$\int_0^d I_{[x,d-y]}(t)dt = 0 = (d-x-y)^+.$$

Therefore, in either cases the inequality $\int_0^d I_{[x,d-y]}(t)dt = (d-x-y)^+$ holds. Thus if $\mathcal{P}$ is the probability measure on the sample space Ω, where X and Y act, then

$$
\begin{aligned}
E[(X+Y-d)^+] &= \int_{\Omega} (d-x-y)^+ dP \\
&= \int_{\Omega}\int_{0}^{d} I_{[x,d-y]}(t)\,dt\,dP \\
&= \int_{0}^{d}\int_{\Omega} I_{[x,d-y]}(t)\,dP\,dt \qquad \text{Fubbini's Theorem} \\
&= \int_{0}^{d} \Pr\,[X \le t \le d-Y]dt \\
&= \int_{0}^{d} \Pr\,[X \le t, Y \le d-t]dt \\
&= \int_{o}^{d} F_{X,Y}(t,d-t)\,dt \qquad (*)
\end{aligned}
$$

0-2

But we have,

$$
\begin{aligned}
(X+Y-d)^+ &= X+Y-d+(X+Y-d)^- \\
&= X+Y-d+(d-X-Y)^+
\end{aligned}
$$

so then by taking the expectation and using the result of equation (0-2), we can obtain

$$E[(X+Y-d)^+] = E(X)+E(Y)-d+\int_{0}^{d} F_{X,Y}(t,d-t)dt$$

■

Definition (Stop-loss order)

A risk X is said to precede a risk Y in stop-loss order, written $X \le_{sl} Y$, if for all retentions $d \ge 0$, the net stop-loss premium for risk X is smaller than that for risk Y:

$$E[(X-d)^+] = E[(Y-d)^+]$$

Theorem

For every two random variable X and Y and for the uniformly distributed $U \sim Uniform(0,1)$, we have

$$X+Y \le_{SL} F_X^{-1}(U)+F_Y^{-1}(U)$$

Proof:

We must show that for all d,

$$E[(X+Y-d)^+] = E[(F_X^{-1}(U)+F_Y^{-1}(U)-d)^+]$$

To show this, one first notes that from lemma (2.1.1) both random variables X and $F_X^{-1}(U)$ have the same distribution. Then, $E(X) = E[\,F_X^{-1}(U)]$ and $E(Y) = E[\,F_Y^{-1}(U)]$. Further, from proposition (2.1.3), we have

$$
\begin{aligned}
F_{X,Y}(t,d-t) &\le F_{F_X^{-1}(U),F_Y^{-1}(U)}(t,d-t) \\
\int_{0}^{d} F_{X,Y}(t,d-t)dt &\le \int_{0}^{d} F_{F_X^{-1}(U),F_Y^{-1}(U)}(t,d-t)\,dt
\end{aligned}
$$

Now the theorem follows from lemma (2.1.4).

■

Theorem (Sub-additivity)

For every two random variable X and Y and for the premium principle $H = H_g$, we have

$$H_g[X+Y] \leq H_g[X] + H_g[Y]$$

Proof:

Since H preserves the stop-loss ordering and since H is additive on comonotone risks, we can write

$$\begin{aligned} H[X+Y] &\leq H[F_X^{-1}(U) + F_Y^{-1}(U)] \\ &= H[F_X^{-1}(U)] + H[F_Y^{-1}(U)] \\ &= H[X] + H[Y] \qquad \text{as } X \text{ is distributed identically as } F_X^{-1}(U) \end{aligned}$$

■

Corollary

For any random variable U, uniformly distributed on $[0,1]$, and any risks $X_1, X_2, \dots, X_n$, we have

$$\sum_{i=1}^{n} X_i \leq_{SL} \sum_{i=1}^{n} F_{X_i}^{-1}(U)$$

Note that $X_1, X_2, \dots, X_n$ and $F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)$ have the same marginal distribution, while the risks $F_{X_i}^{-1}(U)$, $i = 1, 2, \dots, n$, are mutually comonotone generalized lemma (2.1.2) for n random variables. This corollary states that in the class of all multivariate risks $(X_1, X_2, \dots, X_n)$ with given marginals, the stop-loss premiums of $(X_1 + X_2 + \dots + X_n)$ are maximal if the risks X_i are mutually comonotone.

More generally, one can obtain sub-additivity for more than two random variables. The multivariate case follows immediately by considering the fact that $H[X+Y+Z]$ can be sub-additive with respect to $H[X+Y]$ and $H[Z]$. In the same way, $H[X_1 + X_2 + \dots + X_{n-1} + X_n]$ is sub-additive respecting $H[X_1 + X_2 + \dots + X_{n-1}]$ and $H[X_n]$. Thus,

$$H[X_1 + X_2 + \dots + X_{n-1} + X_n] \leq \sum_{i=1}^{n} H[X_i]$$

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Turnover tax and stock return volatility: recent experiences from Asia-Pacific countries

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Abstract.
Global financial turmoil in recent years has resulted in renewed interest in taxing financial markets. Advocates believe that the tax would penalize noise trading and thus contribute to the stability of the market, while opponents argue that the tax would hurt stability, by penalizing noise traders and rational traders equally. In view of these policy and academic discourses, concentrating on stock trading, this paper empirically examines whether levying a turnover tax would increase or reduce the price return volatility. To do so, given the increasing importance of Asia-Pacific countries in the world economy, we investigate recent tax reform episodes in those countries, using a GARCH and its variant models, as most of the advanced economies already abolished the tax many years ago. Overall, the estimation yields evidence that the turnover tax did not reduce volatility, in line with findings of earlier studies based on more advanced economies.

Keywords: turnover tax, taxation on financial transactions, stock trading volume, Japanese market

Introduction

Global financial turmoil in recent years has resulted in renewed interest in taxing the financial markets. As the world's leaders gathered in summit meetings to discuss how to prevent the recurrence, some have openly talked about the Tobin tax[1]. In response to growing interest at the G20 meetings, in 2010 IMF published a report to suggest a few new taxes for policy consideration[2]. A number of academics echoed these developments, by advocating the introduction of a turnover tax in the securities market[3].

Interest among policymakers and academics in transaction taxes on securities in general, and stock trading in particular, is by no means new; its history dates back to at least the late 1980s. In the aftermath of Black Monday a debate occurred in the United States about whether the country should reintroduce a stock transaction tax (hereinafter, referred to as STT), which had been abolished several years prior. Advocates claimed that the tax would stabilize the market by pushing destabilizing noise traders out of the market due to the increased transaction cost, and that the tax would also raise large revenues [e.g. Stiglitz (1989), Summers and Summers (1989)]. This is referred to as the conventional view. However, those opposing the tax argued that a higher transaction cost would adversely affect rational traders equally or even more than noise traders, thereby destabilizing rather than stabilizing the market, and that the tax elasticity could be large, thereby failing to raise large revenue [e.g. Hakkio (1991), Kupiec (1996)]. This view is referred to as the contrarian view[4].

Naturally, these policy debates prompted empirical research regarding the effect of STTs on return volatility. Roll (1989), for example, examined the issue with cross-country data from 23 countries and found no significant effect. With data from Sweden, Umlauf (1993) concluded that the introduction of a turnover tax in the country did not reduce volatility. Hu (1998) drew similar findings from tax reform episodes of four Asian countries. Lindgren (1994) used data from 14 countries for 11 years and concluded that a turnover tax of above 0.5% increases volatility, while one with a lower rate has no effect. While these studies used the concept of traditional volatility, Saporta and Kan (1997) examined the UK stamp duty in a standard GARCH (generalized auto-regressive conditional heteroskedasticity) model and found no significant effect on volatility[5]. Hayashida and Ono (2010) investigated the issue in the context of the Japanese tax reforms and reached a conclusion that the tax cuts had no effect or possibly reduced, rather than increased, volatility.

Although existing studies seem to support the hypothesis that tax cuts have no or possibly a negative

[1] Such leaders include, for example, Angela Merkel, Nicolas Sarközy and Gordon Brown.
[2] *Global Financial Stability Report: Meeting New Challenges to Stability and Building a Safer System*.
[3] For instance, many economists from all over the United States co-signed "An Open Letter from Economists in Support of Financial Transaction Taxes", in December 2009, which was made public through the Center for Economic Policy Research.
[4] The terms "conventional view" and "contrarian view" are borrowed from Song and Zhang (2006).
[5] Baltagi et al. (2006) examined the stock transaction reform in 1997in China. They used a GARCH-type model but only to examine the market efficiency aspect of the tax; for return volatility, however, they use a method based on the concept of historical volatility to conclude that the increase in the tax rate led to greater volatility.

effect on volatility, *i.e.* the contrarian view, they predominantly address advanced economies and/or are outdated. However, the 'geography' of the world's economy and finance has dramatically changed in the last two decades or so; now, Asia is praised as a world growth center. Given this, the present paper attempts to improve the literature by taking up recent tax reform episodes in five Asia-Pacific countries that still have STTs or have had them until recently: China, Korea, India, Australia, and Hong Kong. In doing so, the paper takes advantage of recent advancement in modeling stock return volatility in finance literature, and employs a GARCH model and its variant.

The paper is organized as follows. The next section explains the STT reforms of the five Asia-Pacific countries. The third section explains the methodology and data. The fourth section reports the results. The final section concludes the paper.

STT reforms in Asia-Pacific countries

China. As China is the world's fastest growing economy, its financial markets, too, have expanded rapidly over the last two decades or so, since a stock exchange was first established in December 1990.[6] As of the end of 2009, its total market value almost paralleled that of the Japanese markets, the second largest in the world (see Figure 1). There are two major stock exchanges in the country, one in Shanghai and the other in Shenzhen. In both exchanges, two types of shares are traded. Type A shares are traded only by domestic investors, while type B shares are traded by both domestic and foreign investors. The trading volume of type A shares is far greater than that of type B shares. The ratio is roughly one to one hundred as of the end of 2009.

China has levied the STT from a very early stage, but has also changed the tax rate quite frequently. Since the mid-1990s, the government changed the rate six times, decreasing it four times and increasing it twice.

Korea. The Korean security market developed in the late 1960s. At the very first stage, the government took a strong lead in its development; for instance, it forced 47 conglomerate-based firms to list their shares. Due to the political instability and concurrent economic turmoil, however, the market experienced a long period of stagnation until the late 1980s. Since then, despite a short period of disruption due to the Asian currency crisis in 1998, the stock market has steadily grown. The total market value increased 7.3 times from 1990 until 2008. As of the end of 2009, the Korea Stock Exchange ranks the thirteenth in the world (see Figure 1).

The STT has been levied since the 1970s. The rate has two layers: a general rate, which has been 0.5% since January 1990, and a temporary rate, which is added onto the general rate and changed from time to time, depending on the market situation. Since the mid 1990s, it was reduced twice (see Table 1).

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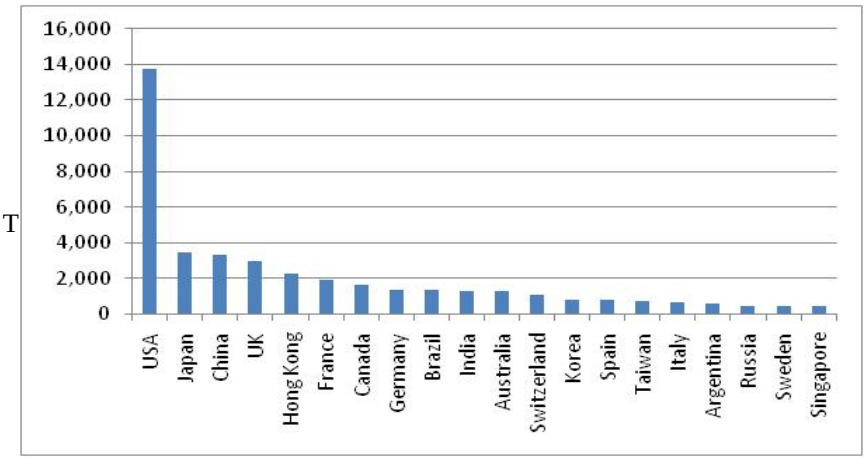


Figure 1. T as of the end of

India.

India's stock market is the oldest in Asia; it dates back to as early as the 1850s. There were more than 20 stock markets in the country, but since the mid-1990s, the transaction has started to concentrate in the largest two markets, the Mumbai (Bombay) and National stock exchanges. Their combined share of the nation's total transactions is now 99.9%.

As the Indian economy boomed in the last decade or so, however, its stock market too rapidly expanded. The Bombay Stock Exchange (BSE) now ranks the tenth in the world in total market value as of the end of 2009 (see Figure 1). In order to cool off the market, the government introduced a STT in October, 2004 and raised it in 2006, while many advanced economies had already abolished the STT (see Table 1).

Australia.

A stock market was first established in Melbourne in 1871, some 10 years after the Gold Rush. Since

[6] See, for example, Japan Securities Research Institute (2010) for greater details about each country's market.

then, many stock exchanges were established all over the country. In 1937, these independent exchanges were unified into the Australian Associated Stock Exchange. After it was reorganized as a limited company, it rigorously promoted online trading. Partly due to such efforts, trading expanded rapidly; as of the end of 2009, its total market value ranks the 11th in the world and the 5th in Asia (see Figure 1). The country abolished its STT altogether in July 2000, following the same actions in other developed countries (see Table 1)

Table 1. Summary of STT reforms in the five countries (after 1995).
Source: Japan Securities research Institute (2010).

Country	Tax cut	Tax increase
China	June 12, 1998, November 16, 2001; January 24, 2005; April 24, 2008	May 12, 1997; May 30, 2007
Korea	July 14, 1995; March 30, 1996	
India		October 1, 2004 (introduction); June 1, 2006
Australia	July 1, 2000 (abolition)	
Hong Kong	April 1, 1998; April 7, 2000; September 1, 2001	

Hong Kong.

As an "entrance" to Asian markets for European traders, Hong Kong's regulations and barriers to trade and investment have long been held minimum, to make it the freest economic entity in Asia. Now, it also plays a role of "entrance" to China in many aspects of its economy. In stock markets, for instance, the total share of China-related stocks in the overall market value now exceeds 60%.

Although the freest market in the region, Hong Kong has been levying a STT. However, it has reduced the rate three times since the mid-1990s (see Table 1). It is now only 0.1%.

Methodology and data

Time-varying volatility model and GARCH.

In empirical investigations of stock return volatility, it is now a standard practice to use a GARCH-type model. In a simple model of historical volatility based on variance and standard deviation calculated from the data, the data-generating process (DGP) of stock return, R_t, is assumed to be unchanged for a certain period of time; so are its mean and variance. However, observed returns are high for quite some time, and then become low for another time. The same is true for observed volatility. This raises doubt about such a simple assumption as a time-invariant DGP.

Assume then that, give the information up to the date t-1, the mean of return at date t, R_t, is no longer constant and varies. Then, R_t can be divided into a part that is forecastable at date $t-1$ $E_{t-1}(R_t)$, and a shock that is not forecastable, ε_t.

$$R_t = E_{t-1}(R_t) + \varepsilon_t$$

This is termed the mean equation. An example of this is the following, which expresses R_t changes over time.

$$R_t = a + \sum_{i=1}^{p} b_i R_{i-p} + \varepsilon_t \quad (1)$$

The unforecastable shock ε_t is expressed as

$$\varepsilon_t = \sigma_t z_t$$
$$\sigma_t > 0$$
$$z_t \sim i.i.d.(0,1)$$

Here, σ_t is assumed to always take a positive value. σ_t^2 is termed as volatility of R_t at date t. Note that σ_t^2 is variable with respect to t.

Further, observed volatility tends to stay high (low) for some time once it becomes high (low). This phenomenon, known as *volatility clustering*, therefore needs to be incorporated into the model. Accordingly, the ARCH(p) model (Engle, 1982) has been invented to consider a long-lasting effect of the shock,

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}.$$

Bollerslev (1986) generalized ARCH (p) to form the GARCH(p,q) model:

$$\sigma_t^2 = \omega + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}.$$

Here, q represents the length of "memory" of past volatility. Note that, because volatility is non-negative, restrictions $\omega>0$, $\alpha_i > 0$, and $\beta_i>0$ apply.

Asymmetric model.

It is a well-known fact that bad news, which brings down the price, has a greater impact on volatility than good news. Within the class of GARCH-type models, a number of variants have been proposed to incorporate this generally observed asymmetry of the return volatility, known as the *leverage effect*. The EGARCH (p, q) model, first proposed by Nelson (1991), is one of the most widely used among the asymmetry models. It considers the following model, taking the logarithm of the volatility and imposing a non-negative constraint on its parameter, in the following manner:

$$\log(\sigma_t^2) = \omega + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2) + \sum_{i=1}^{p} \alpha_i \left(|z_{t-i}| - E\left(|z_{t-i}| \right) \right) + \sum_{l=1}^{m} \gamma_l z_{t-l} \ ,$$

where $z_{t-1} = \varepsilon_{t-1} / \sigma_{t-1}$ and γ is the parameter for asymmetry. In the simplest form of GARCH (1,1) (and m=1),

$$\log(\sigma_t^2) = \omega + \beta_1 \log(\sigma_{t-1}^2) + \alpha_1 |z_{t-1}| - \alpha_1 E\left(|z_{t-1}| \right) + \gamma z_{t-1} \ ,$$

if $z_{t-1} \geq 0$,

$$\log(\sigma_t^2) = \omega + \beta_1 \log(\sigma_{t-1}^2) + \gamma z_{t-1} + (\gamma + \alpha_1) z_{t-1} - \alpha_1 E\left(|z_{t-1}| \right),$$

and if $z_{t-1} < 0$,

$$\log(\sigma_t^2) = \omega + \beta_1 \log(\sigma_{t-1}^2) + \gamma z_{t-1} + (\gamma - \alpha_1) z_{t-1} - \alpha_1 E\left(|z_{t-1}| \right).$$

Therefore, $\gamma<0$ implies the existence of the asymmetry.

In this paper, we employ EGARCH in addition to the GARCH model.

Estimation details.

In what follows, we investigate how the STT reforms discussed in the previous section affected stock return volatility, using the two models discussed above. It is often argued that stock return volatility exhibits a day-of-the-week effect. That is, for example, it is higher on Monday than in the rest of the week. In order to address this possibility, we create day-of-the-week dummies in the following manner, and add them in the regression.

$$WD_t^k = \begin{cases} 0 & if\ t\ fallsonthekthdayoftheweek \\ 1 & if\ otherwise \end{cases}, \quad k = 1, 2, \ldots 4$$

Here k=1 corresponds to Monday. Note that k is 1 through 4; that is, Friday is the base line of the regression. We will try specifications both with and without the day-of-the week effect.

To examine the effect of tax reforms, we add a dummy variable representing the reform, which takes the values of 0 and 1, respectively, before and after the date when the change was put into effect. To examine the effect of a STT, we represent it by:

$$D_t = \begin{cases} 0 & if\ t \leq t_0 \\ 1 & if\ otherwise \end{cases},$$

To summarize, the mean equation with the day-of-the-week effect takes the form:

$$R_t = a + bR_{t-1} + dD_t + \sum_{k=1}^{4} c_k WD_t^k + \varepsilon_t \ .$$

The variance equation with the day-of-the week effect, in the case of the EGARCH, is

$$\log(\sigma_t^2) = \omega + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2) + \sum_{i=1}^{p} \alpha_i \left(|z_{t-i}| - E\left(|z_{t-i}| \right) \right) + \gamma z_{t-1} + \delta D_t + \sum_{k=1}^{4} \theta_k WD_t^k + \lambda TV_t \ .$$

As mentioned earlier, we employ two specifications for each of the tax reforms. In Specification I, we restrict c_k and θ_k, to be zero (k = 1, 2, 3, and 4), whereas these restrictions are relaxed in Specification II.

Specification I: $c_k = \theta_k = 0$ (without day-of-the-week effect)

Specification II: $f_k \neq 0$, $\theta_k \neq 0$ (with day-of-the week effect)

In the analysis below, we select the values of p and q by AIC, changing them each from 1 to 3.

The data used are the most popular aggregate price indices for each market: Shanghai Stock Exchange A share (CHASHR) for China, Korea Stock Exchange Composite (KOSPI) for Korea, India BSE (100) National (IBOMBSE) for India, Standard and Poor's/Australian Stock Exchange 200 (ASX201) for Australia, and HANG SENG PRICE INDEX (HNGKNGI) for Hong Kong. All daily closing data are taken from Thompson Reuters' Datastream . For the estimation period, we take 6 months before and after the date of the tax change. Table 2 shows the descriptive statistics of the data used in the estimation below.

Table 2. Descriptive statistics for aggregate price return in the five countries (after 1995)

country	China						Korea
date of tax change	1997/5/12	1998/6/12	2001/11/16	2005/1/24	2007/5/30	2008/4/24	1995/7/14
maximum	0.09481	0.04981	0.093998	0.079014	0.051979	0.08888	0.042278
minimum	−0.104468	−0.08798	−0.065053	−0.0396	−0.092608	−0.08045	−0.039493
mean	0.000924	0.000241	−0.001448	−0.001216	0.003534	−0.004893	−0.000433
standard dev.	0.027775	0.013275	0.016799	0.014573	0.022588	0.026515	0.011393
no. of observations	244	247	239	242	244	224	244
country	Korea	India		Australia	Hong Kong		
date of tax change	1996/3/30	2004/10/1	2006/6/1	2000/7/1	1998/4/1	2000/4/7	2001/9/1
maximum	0.027976	0.137049	0.053373	0.024598	0.172471	0.054342	0.043454
minimum	−0.036648	−0.128047	−0.071731	−0.055498	−0.147347	−0.08939	−0.092854
mean	−0.000911	0.003497	0.00071	0.000111	−0.002618	0.000878	−0.001433
standard dev.	0.011042	0.019138	0.019076	0.008917	0.032968	0.019337	0.017447
no. of observations	245	250	250	252	247	248	244

Estimation results.

Table 3 (at the end of this paper) exhibits the results for GARCH. It is clear that, in many cases, the coefficients for the tax dummy, D, are insignificant. This is true whether the change is a tax cut or an increase. When significant, the sign of the coefficient is in the same direction as the change in the tax rate, supporting the contrarian view. This is in line with the existing literature regarding the more advanced economy. The only notable exception is the tax increase in June 2006 in India. The coefficient for D is of the negative sign and significant at the 1% level. This suggests that the tax increase led to a reduction in volatility, in line with the conventional view.

Table 4 (at the end of this paper) exhibits the results for EGARCH. The coefficient of asymmetry, γ, is largely negative, although not uniformly significant, suggesting a leverage effect. The sign of the coefficient for D is largely in the same direction as the change in the tax rate. However, besides the tax increase in 2006 in India, different results are obtained in a few cases: (a) the tax cut in June 2000 in China (specifications I and II) , (b) the tax cut in Korea (specification I) in 1994, and (c) the tax cut in 1996 in Hong Kong (specification II). However, for (b), the coefficient is insignificant in Specification II with the dummy for Monday being significant. In (c), the significance is only at the 10% level. Only (a) may be taken as decent evidence for the conventional view[7]. In 21 other cases out of 28, however, the obtained results are in line with the contrarian view or suggestive that STT has no effect on volatility, as found in GARCH estimations in Table 3 and existing literature based on more advanced economies.

Conclusion

While global financial turmoil in recent years has resulted in renewed interest in taxing financial markets, there have been two opposing views on the effect of turnover tax on return volatility. In the conventional view, the tax reduces volatility, whereas it increases volatility in the contrarian view. The existing literature either suggests that the tax has no effect or else supports the contrarian view, but it predominantly addresses advanced economies and/or is outdated.

The world's 'geography' of finance has significantly changed since the days when those advanced economies levied STTs; now many countries in the Asia-Pacific region rank among "financial powers." As these countries still have STTs or have had them until recently, this paper took up episodes of STT reform in those countries, and investigated the effects of STTs using GARCH and EGARCH models. The results obtained are in line with the existing literature based on more advanced economies: the STT has no effect or possibly an effect to reduce volatility, but never to increase it.

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Table 3. Estimation results for GARCH

Country		China												Korea	
Date of tax change		1997/5/12		1998/6/12		2001/11/16		2005/1/24		2007/5/30		2008/4/24		1995/7/14	
Nature of tax change		Tax increase		Tax cut		Tax cut		Tax cut		Tax increase		Tax cut		Tax cut	
Specification		Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ
mean equation	c	3.79	4.39	1.44	1.20	−0.44	−2.35	−2.086**	3.88	8.098***	12.82***	−5.259***	−3.42	−0.16	−0.51
	rr(−1)	−42.81	−69.39	112.96	65.37	−23.39	53.67	−13.33	−46.60	−85.688*	−24.08	−104.672**	−108.39	139.876***	117.10
	tax	−3.82	−7.488**	−1.82	−1.77	−0.12	0.93	0.33	−2.22	−6.82**	−7.792**	−0.70	−2.71	−0.95	0.41
	mon	−	−0.05	−	0.11	−	0.22	−	−6.918***	−	−8.133*	−	−4.01	−	−0.97
	tue	−	4.99	−	−1.68	−	5.325**	−	−6.83**	−	−0.08	−	−0.71	−	−2.19
	wed	−	5.82	−	−0.17	−	−1.76	−	−4.60	−	−2.15	−	−0.97	−	1.86
	thu	−	−1.47	−	1.74	−	2.42	−	−0.59	−	−8.742*	−	−2.13	−	0.17
variance equation	c	0.215***	0.102**	0.0773***	0.01	0.01	0.123***	0.08	0.15	1.423***	0.381***	1.234***	1.377**	0.161***	0.0759*
	tax	−0.06	−0.01	0.00	0.037*	0.01	0.00	0.05	0.02	0.692***	0.08	0.37	0.10	0.00	0.02
	mon	−	−0.03	−	0.0755**	−	−0.141***	−	−0.168*	−	0.232*	−	−0.24	−	0.12***
	tue	−	−0.237***	−	0.04	−	−0.135***	−	−0.05	−	−0.12	−	−0.593***	−	−0.07
	wed	−	−0.04	−	0.091**	−	−0.122**	−	−0.05	−	0.12	−	−0.365***	−	−0.02
	thu	−	0.05	−	0.124***	−	−0.0783*	−	−0.08	−	0.198*	−	−0.252*	−	−0.03
order	p	1	3	1	1	3	2	3	3	3	3	2	3	3	3
	q	1	2	1	2	3	2	3	1	3	3	1	3	3	1
Adj. r−squared		0	0	0	0	0	0	0	0	0	0	0	0	0	0
AIC		−4.47	−4.51	−5.95	−5.92	−5.59	−5.59	−5.68	−5.65	−4.78	−4.78	−4.42	−4.40	−6.18	−6.17
no. of observation		244	244	247	247	239	239	242	242	244	244	224	224	244	244

Country		Korea		India				Australia		Hong Kong					
Date of tax change		1996/3/30		2004/10/1		2006/6/1		2000/7/1		1998/4/1		2000/4/7		2001/9/1	
Nature of tax change		Tax cut		Tax increase		Tax increase		Tax cut (abolition)		Tax cut		Tax cut		Tax cut	
Specification		Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ
mean equation	c	−0.62	1.67	2.525**	8.491**	0.67	3.831*	1.11	0.33	−0.47	8.38	2.10	0.59	−2.20	−2.51
	rr(−1)	174.583**	206.212***	161.364*	190.856**	187.417***	200.069***	24.78	−0.84	54.47	70.12	76.34	96.95	−53.76	18.16
	tax	−0.23	−2.997**	1.74	1.34	1.39	0.22	−1.43	−1.95*	−4.20	−4.49	−1.92	−1.41	3.31	1.38
	mon	−	−5.009***	−	−7.644*	−	−7.805***	−	1.56	−	−7.36	−	3.16	−	1.13
	tue	−	0.68	−	−9.427**	−	−2.00	−	2.284*	−	−9.09	−	0.95	−	2.24
	wed	−	1.75	−	−0.76	−	−5.923**	−	1.00	−	−19.108***	−	0.66	−	−3.40
	thu	−	−3.601*	−	−6.997*	−	0.05	−	−0.23	−	−5.75	−	0.69	−	−1.72
variance equation	c	0.00	0.08	0.0144**	0.354***	0.00714***	0.281***	0.0414**	0.0425***	0.174**	0.627***	0.00772***	0.07	0.02	0.02
	tax	0.00	0.02	0.0418**	0.03	−0.0063***	−0.199***	−0.0244*	−0.0149*	−0.02	−0.01	−0.00479***	−0.01	0.00	−0.0133**
	mon	−	0.00	−	−0.325***	−	0.11		0.00	−	−0.503***	−	0.00	−	−0.09
	tue	−	−0.03	−	−0.27***	−	0.05		−0.0405**	−	−0.703***	−	−0.185***	−	0.01
	wed	−	0.01	−	0.07	−	0.0992*		−0.0503***	−	−0.604***	−	0.11	−	0.168*
	thu	−	−0.04	−	−0.385***	−	0.04		0.00	−	0.05	−	−0.231***	−	0.02
order	p	2	3	2	1	2	1	2	2	3	2	1	2	3	1
	q	2	2	2	1	1	2	2	2	2	2	1	1	1	2
Adj. r−squared		0.02	0.02	−0.09	−0.08	0.02	0.00	−0.01	−0.02	−0.02	0.00	0.00	−0.01	−0.01	−0.01
AIC		−6.18	−6.21	−5.39	−5.42	−5.29	−5.21	−6.75	−6.76	−4.21	−4.21	−5.08	−5.09	−5.31	−5.29
no. of observation		245.00	245	250	250	248	248	252	252	247	247	248	248	244	244

Notes: *, **, and *** indicate that the reported coefficient is significant at the 10, 5, and 1% level, respectively.

Table 4. Estimation results for EGARCH

Country		China												Korea	
Date of tax change		1997/5/12		1998/6/12		2001/11/16		2005/1/24		2007/5/30		2008/4/24		1995/7/14	
Nature of tax change		Tax increase		Tax cut		Tax cut		Tax cut		Tax increase		Tax cut		Tax cut	
Specification		Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ
mean equation	c	1	4	0	1	0	-3.504**	-1	-2	8.721***	9.055***	-4.036***	-4	0	0
	rr(-1)	75.342*	-132.499**	110.204**	52	75	138.029**	-65	-193.942***	123.949***	73.012***	-89.756***	-61	126.322**	98.776*
	tax	1	-6.704***	-1	-1	-2.459**	-3.528***	-2	2.268*	-10.66***	-7.419***	-2	0	-1.328**	1
	mon	-	0	-	0	-	3	-	-2	-	-4.932**	-	-4	-	-2
	tue	-	4	-	-1	-	4.789*	-	-2	-	0	-	1	-	-3.184*
	wed	-	5	-	-1	-	2	-	0	-	7.948***	-	-4	-	0
	thu	-	1	-	1	-	4	-	-4.133**	-	-5.291**	-	-1	-	0
variance equation	c	-2675.385***	-2148.517***	-3681.554**	-2825	-342.415***	-77	-3493	-8837.458***	-5508.95***	-10910.28***	-17237.93***	-13864.66***	-545.997***	-2434
	tax	-84	-4	16	-24	81.079***	77.084***	51	40	801.963***	986.314***	223	210	37.644***	-9
	mon	-	-1025.199***	-	-82	-	-502	-	-750	-	1547.854***	-	41	-	731.087*
	tue	-	-1016.272***	-	103	-	286	-	68	-	438.049***	-	79	-	-465
	wed	-	-1441.905***	-	-15	-	-862	-	-203	-	984.146***	-	25	-	-206
	thu	-	-250	-	-243	-	335	-	-455	-	1149.838***	-	139	-	238
γ		-137.417***	-95	-189.751***	-199.999***	-237.229***	-167.61***	-40	88	-725	-436.366***	176.461***	188	-170.511***	-226.491**
order	p	3	1	1	2	3	3	1	2	2	3	3	3	2	1
	q	1	3	2	2	1	2	3	3	1	3	3	2	2	2
Adj. r-squared		-0.00411	-0.00852	-0.0005	-0.00517	-0.02597	-0.04998	-0.01859	-0.07664	-0.03582	-0.02881	-0.0098	-0.03229	0.003513	-0.00572
AIC		-4.85914	-4.59269	-6.03162	-6.00705	-5.75907	-5.73718	-5.68878	-5.77088	-4.82596	-5.00244	-4.50818	-4.53858	-6.2864	-6.18468
no. of observation		244	244	247	247	239	239	242	242	244	244	224	224	244	244

Country		Korea		India				Australia		Hong Kong					
Date of tax change		1996/3/30		2004/10/1		2006/6/1		2000/7/1		1998/4/1		2000/4/7		2010/9/1	
Nature of tax change		Tax cut		Tax increase		Tax increase		Tax cut (abolition)		Tax cut		Tax cut		Tax cut	
Specification		Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ	Ⅰ	Ⅱ
mean equation	c	-1.157**	-0.419	5.549***	5.22**	0.584	2.375	0.282	0.401	-3.197	5.991	1.046	-2.026	-2.207***	-1.797
	rr(-1)	222.487***	147.731***	-19.853	126.034*	183.507***	199.853***	39.737	72.151	74.765	40.87	129.377**	156.116***	0.331	6.598
	tax	4.33E-01	-8.07E-01	-0.879	0.017	1.385	2.087	-0.319	-0.0423	-4.654*	-7.231**	-2.097	-1.277	1.896	0.93
	mon	-	-3.098*	-	-3.249*	-	-7.977**	-	0.487	-	-5.528	-	4.035	-	0.389
	tue	-	1.436	-	-4.634*	-	-2.11	-	-0.444	-	-8.593**	-	1.186	-	2.227
	wed	-	-0.144	-	-1.125	-	-3.598	-	-0.986	-	-15.814***	-	-1.656	-	-3.208
	thu	-	0.485	-	-4.849**	-	0.466	-	-2.492*	-	-6.202	-	4.796	-	-0.272
variance equation	c	-2404.617*	-2775.804	-7604.222***	-6274.097**	-42.178	-271.469	-7538.063***	-8203.456*	-145.59***	-180.777	-685.156**	-1327.034***	-443.57	-238.631
	tax	45.665	120.16	838.612***	683.769*	-32.823***	-59.966**	-823.517**	-0.919593**	38.131*	54.346*	-14.006	-8.354	7.958	9.102
	mon	-	-625.975*	-	-1684.259***	-	107.007	-	45.661	-	316.864	-	1014.681**	-	-286.332
	tue	-	-99.304	-	-447.838**	-	-205.086	-	-389.301	-	-772.169**	-	223.853	-	-206.162
	wed	-	-367.902	-	-1142.296***	-	394.661	-	-202.54	-	216.034	-	910.338*	-	391.822
	thu	-	-382.572	-	-1243.885***	-	251.139	-	-74.812	-	182.113	-	960.308*	-	58.512
γ		-157.292	-163.321	-34.615***	38.315	-33.003	-76.076	-89.311**	-57.771	-243.222***	-263.991	-285.668***	-258.706***	-131.276***	-131.638*
order	p	2	3	3	3	3	3	2	2	1	1	1	1	1	1
	q	3	3	3	3	1	2	2	2	1	1	3	2	2	2
Adj. r-squared		0.013381	0.01631	-0.01138	-0.04929	0.018446	0.011089	-0.00912	-0.01772	-0.03136	-0.00684	0.000451	0.005138	-0.00564	-0.00828
AIC		-6.23324	-6.21873	-5.65691	-5.64203	-5.27069	-5.24253	-6.85001	-6.83112	-4.34882	-4.37616	-5.1445	-5.11663	-5.34119	-5.3298
no. of observation		245	245	250	250	248	248	252	252	247	247	248	248	244	244

Notes: *, **, and *** indicate that the reported coefficient is significant at the 10, 5, and 1% level, respectively.

A note on the time evolution of bivariate copulas

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Abstract.
Dependence properties between random events are important issues not only for researches but also from the viewpoint of risk management. For example, the dependence relation between exchange rates is worth further investigation both for academics and for insurance industries. To discuss such association problems, Kendall's tau τ and Spearman's rho ρ are known measures of concordance between random variables. On the other hand, a copula is well employed tool for analyzing dependence relations, and the formula of τ and ρ in terms of copula is already known. In our previous study, we introduce the time evolution of bivariate copulas, which is provided as a solution of diffusion equation. This paper is concerned with the convergence in time of τ and ρ under the evolution of copulas. It is shown that any dependence structure converges to the one of independence as the time tends to infinity.
Key words: copula, bivariate copula, risk management

Introduction

Dependence relations between random variables are one of the most important subjects for researches in probability and statistics. For example, the dependence between exchange rates, such as the rate between Japanese yen (JY) against US dollar and JY against Euro is worth intensive investigation. To mention another, the damage caused by typhoon is randomly related between Taiwan and Japan; some typhoon move from Taiwan to Japan but others do not. The insurance company whose business includes these two countries wants to estimate the effect of this dependence. To quantitatively estimate such dependences, several measures of associations have introduced so far. As widely known examples, we recall the population version of Kendall's tau and Spearman's rho, whose explicit formulation will be provided shortly later.

On the other hand, a copula is introduced as a tool for understanding the dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions. The definition of copula and the fundamental theorem due to A. Sklar [6] is expressed as follows in the case of a bivariate joint distribution.

Definition 1. A function C defined on $I^2 := [0,1] \times [0,1]$ and valued in I is called a copula if the following conditions are fulfilled.

For every $(u,v) \in I^2$

$$C(u,0) = C(0,v) = 0, \quad C(u,1) = 1 \text{ and } C(1,v) = v. \tag{1}$$

For every $(u_i, v_i) \in I^2 (i = 1,2,)$ with $u_1 \le u_2$ and $v_1 \le v_2$

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \ge 0 \tag{2}$$

The second requirement is referred to as the 2-increasing condition. We also note that a copula is continuous by its definition

Theorem 2. (Sklar's theorem) Let H be a bivariate joint distribution function with marginal distribution functions F and G; that is,

$$\lim_{x\to\infty} H(x,y) = G(y), \qquad \lim_{y\to\infty} H(x,y) = F(y),$$

Then there exists a copula, which is uniquely determined on $RanF \times RanG$, such that

$$H(x,y) = C(F(x), G(y)). \tag{3}$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (3) is a bivariate joint distribution function with marginals F and G..

The population version of Kendall's tau and Spearman's rho, which are denoted by τ and ρ, respectively, are known to be represented in terms of copulas. See for instance an excellent monograph by R.B. Nelsen [4]. Precisely stated, let X and Y be continuous random variables whose copula is C. Then we have

$$\tau_{X,Y} = \tau_C = 4\iint_{I^2} C(u,v)\,dC(u,v) - 1 = 1 - 4\iint_{I^2} \frac{\partial C}{\partial u}(u,v)\frac{\partial C}{\partial v}(u,v)\,du\,dv$$

$$\rho_{X,Y} = \rho_C = 12\iint_{I^2} C(u,v)du\,dv - 3 = 12\iint_{I^2} (C(u,v) - uv)du\,dv$$

Now, in our previous study [3], we have introduced the time evolution of copulas. That is, we consider a time parameterized family of copulas $\{C(u,v,t)\}_{t\geq 0}$, which satisfy the heat equation:

$$\frac{\partial C}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)C(u,v,t). \tag{4}$$

Here, by the definition of copula, we understand that $C(\cdot, \cdot, t)$ fulfills (1)(2); to be precisely, we postulate that:

1. For every $(u,v,t) \in I^2 \times (0,\infty)$,

$$C(u,0,t) = C(0,v,t) = 0, \qquad C(0,1,t) = u, \qquad C(1,v,t) = v. \tag{5}$$

2. For every $(u_i, v_i, t) \in I^2 \times (0,\infty)(i = 1,2,)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$

$$C(u_1,v_1,t) - C(u_1,v_2,t) - C(u_2,v_1,t) + C(u_2,v_2,t) \geq 0. \tag{6}$$

The stationary solution to (4), which is referred to as the harmonic copula, is uniquely determined to be $\Pi(u,v) := uv$ in view of the boundary condition (1). We note that the copula Π represents the independent structure between two respective random variables.

The primal establishment of [3] is the existence proof of solutions to (4), which satisfy the copula conditions (5)(6). In particular, the solution u = u(u, v, t) is expressed as

$$C(u,v) = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \iint_{I^2} \sin m\pi\xi \sin n\pi\eta (C_0(\xi,\eta) - \xi\eta d\xi d\eta \tag{7}$$

In this article, we are concerned with the convergence as $t \to \infty$ of solutions to (4). Since $C = C(u,v,t)$ verifies the diffusion equations, it is natural to expect that C convergex exponentially as $t \to \infty$ in some function spaces. Indeed, we explicitly show the exponential convergence of C as well as τ_C and ρ_C

Now the main theorem of this paper reads as follows.

Theorem 3. Let be a time-parametrized family of bivariate copulas, which satisfy

$$\frac{\partial C}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)C(u,v,t). \text{ for } (u,v,t) \in I^2 \times (0,\infty), \tag{8}$$

$$C(u,v,0) = C_0(u,v) \quad \text{on } (u,v) \in I^2$$

where C_0 denotes a given initial copula. Then it follows that

$$|\tau_{C_t}| + |\rho_{C_t}| \leq Ae^{-Bt}$$

as $t \to \infty$ where A and B are positive constants.

In particular, $C_t \to \Pi$ exponentially as $t \to \infty$.

In the next section, we prove this theorem. Discussions are given in the final section

Proof.

We employ the notation: $\|(C-\Pi)(t)\|_2^2 := \iint_{I^2} (C(u,v,t) - uv)^2 du\,dv$. The proof proceeds with elementary estimates.

We first compute

$$\frac{1}{2}\frac{d}{dt}\|(C-\Pi)(t)\|_2^2 = \iint_{I^2} (C-\Pi)\Delta(C-\Pi)du\,dv = -\|\nabla(C-\Pi)(t)\|_2^2 \leq -\alpha\|(C-\Pi)(t)\|_2^2 \tag{9}$$

where $\alpha > 0$ denotes a constant due to the Poincaré inequality. The use of this inequality is

possible thanks to. We thus obtain

$$\|(C-\Pi)(t)\|_2^2 \le \|(C_0-\Pi)\|_2^2 e^{-2\alpha t}$$

Taking into account of this estimate, we infer that

$$\left|\rho_{C_t}\right| = \left|12\iint_{I^2}(C-\Pi)du\,dv\right| \le 12\|(C_0-\Pi)\|_2 e^{-\alpha t}$$

This proves the exponential convergence of ρ .

Next we turn our attention to τ . We recall the next formula, which is derived upon direct calculation.

$$\tau_c = \frac{2}{3}\rho_c - 4\iint_{I^2}\frac{\frac{\partial}{\partial u}(C(u,v)-uv)}{\frac{\partial}{\partial v}(C(u,v)-uv)}\frac{\partial}{du}dv$$

It then follows that

$$e^{\alpha t}|\tau_C(t)| \le \frac{2}{3}e^{\alpha t}|\rho_C(t)|$$
$$\le 8\|(C_0-\Pi)\|_2 + 4e^{\alpha t}\|\nabla(C-\Pi)(t)\|_2^2$$

In light of the continuity of $\|\nabla(C-\Pi)(t)\|_2$ with respect to t, it suffices to show that

$$\int_t^\infty e^{\alpha s}\|\nabla(C-\Pi)(s)\|_2^2\,ds < \infty$$

which implies that $e^{\alpha t}\|\nabla(C-\Pi)(t)\|_2^2 \le M < \infty$ for all large t with some constant M.

In view of (9), we infer that

$$\frac{\alpha}{2}e^{\alpha t}\|(C-\Pi)(t)\|_2^2$$
$$\le \frac{\alpha}{2}e^{-\alpha t}\|(C_0-\Pi)\|_2^2 ,$$

from which we conclude that

$$2\int_t^\infty e^{\alpha s}\|\nabla(C-\Pi)(s)\|_2^2\,ds \le e^{\alpha t}\|(C-\Pi)(t)\|_2^2 + \int_t^\infty \alpha e^{-\alpha s}\|(C_0-\Pi)\|_2^2\,ds$$
$$\le 2\|(C_0-\Pi)(s)\|_2^2 e^{-\alpha t} .$$

In summary, we have established $e^{\alpha t}|\tau_C(t)| \le 8\|(C_0-\Pi)\|_2 + 4M$ and the proof of the exponential convergence of τ is thereby completed.

Discussions

We have established the convergence of time-dependent family of copulas $\{C_t\}_{t\ge 0}$, which satisfy the heat equation. It is shown that C_t converges exponentially to the product copula Π as the time tends to infinity; the corresponding Kendall's tau τ_{C_t} and the Spearman's rho ρ_{C_t} converge exponentially to zero as $t \to \infty$. Since our copulas verify the diffusion equation, dependence relation converges to the standard one, namely, the independence relation.

Copulas are well investigated in these days. Because of their flexible structure, copulas have been applied in many situations. We refer for instance to Darsow et al. (1992) and E.W. Frees and E.A. Valdez [2]. For other materials, we refer to the well summarized review articles of H. Tsukahara [7] and Y.Yoshizawa [8][9].

Time dependent copulas also have been studied so far, which is introduced by A.J. Patton [5] and is referred to as dynamic copulas. See also [10]. Typical procedure of constructing dynamic copulas is as follows: There exists a class of one-parameter families of copulas, known as the Archimedean copulas (see [4]). We take one such copula and denote it by $C(u,v;\rho)$, where ρ is a parameter whose value should belong to some interval $J(\subset \mathbb{R})$. The dynamic copula is then provided as

$$C(u,v;\rho_t) \text{ with } \rho_t = \Lambda(X_t(\rho_{t-1}))$$

where $X_t(\rho_{t-1})$ means some time-series model, say ARMA(p, q)-type process, and Λ is the transformation function which is designed to keep $\rho_t \in J$.

Compared to the concept of dynamic copulas, in our time-dependent model, copula itself evolves according to the heat equation. Our evolving copulas eventually converge to the sta- tionary copula, namely, the product copula. Such phenomena are rather commonly observed in diffusion processes.

From practical point of view, the maturity condition is much appropriate than the initial value setting as we did here. As an example of this version, we recall the next theorem which is exhibited in [3]

Theorem 4. For any bivariate copula $C_T(u,v)$, where T (> 0) denotes the maturity, there exists a unique family of time-parametrized bivariate copulas $\{C(u,v,t)\}_{0\le t\le T}$ such that

$$\frac{\partial C}{\partial t}(u,v,t)+\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t)=0$$

$$C(u,v,T)=C_T(u,v)$$

on

$(u,v)\in I^2$.

Here the convergence property as $t\to -\infty$ is not so relevant issue from the financial standpoint.

Acknowledgement.

The first author (NI) is supported in part by the grant from the Japan Society for the Promotion of Sciences (No.21540117).

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Stoynov process – definition and simulations

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Abstract.
Stoynov distribution is presented and a new stochastic process is defined based on this distribution - Stoynov process. Simulation and graphics for both the distribution and the process in R languages are done.

Key words. Stoynov distribution, Stoynov process, simulation

.

Stoynov distribution

We say that a random variable ξ has Stoynov distribution with parameters n and β and denote this fact $\xi \in ST(n,\beta),$ if the probability mass function of ξ is given by the formula

$$f_{\xi}(x) = \begin{cases} C(n,\beta)e^{-\beta x}(1+x)^n, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Here $C(n,\beta)$ are suitable coefficients. For example, the special case $C(0,\beta) = \beta$ corresponds to the case of exponential distribution.

We may present the density $f_{\xi}(x)$ for $x \geq 0$ in the following way:

$$C(n,\beta)e^{-\beta x}(1+x)^n = C(n,\beta)e^{-\beta x}\sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}C(n,\beta)\binom{n}{k}x^k e^{-\beta x} =$$

$$= \sum_{k=0}^{n}\frac{C(n,\beta)n!x^k e^{-\beta x}}{k!(n-k)!} = \sum_{k=0}^{n}\frac{C(n,\beta)n!\beta^{k+1}x^k e^{-\beta x}}{\beta^{k+1}(n-k)!k!} =$$

$$= \sum_{k=0}^{n}\frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}\frac{\beta^{k+1}x^k e^{-\beta x}}{k!} =$$

$$= \sum_{k=0}^{n}P(D^n = k+1)f_{G^{k+1}}(x) =$$

$$= \sum_{k=1}^{n+1}\frac{C(n,\beta)n!\beta^k x^{k-1}e^{-\beta x}}{\beta^k(n-k+1)!(k-1)!} =$$

$$= \sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^k(n-k+1)!}\frac{\beta^k x^{k-1}e^{-\beta x}}{\Gamma(k)} = \sum_{k=1}^{n+1}\frac{C(n,\beta)n!}{\beta^k(n-k+1)!}\frac{\beta^k x^{k-1}e^{-\beta x}}{(k-1)!} =$$

$$= \sum_{k=1}^{n+1}f_{D^x,\xi}(k,x) = \sum_{k=1}^{n+1}P(D^n = k)f_{\xi}(x \mid D^n = k) = \sum_{k=1}^{n+1}P(D^n = k)f_{G^k}(x)$$

where D^n are random variables taking values $k = 1,\ldots,(n+1)$ with probabilities $P(D^n = k) = \frac{C(n,\beta)n!}{\beta^k(n-k+1)!}, k = 1,\ldots,(n+1),$ and the random variables $G^k \in \Gamma(k,\frac{1}{\beta}) \equiv Erlang(k,\frac{1}{\beta}),$ i. e. $\xi \mid D^n \equiv Erlang(D^n,\frac{1}{\beta}).$

Also, variables $\widetilde{D}^n = D^n - 1$ can be introduced taking values $k = 0,\ldots,n$ with probabilities

$$P(\widetilde{D}^n = k) = \frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!}, k = 0,\ldots,n.$$

ST(n,β) distribution in the case $n=2$ is introduced in Stoynov (2010) and generalized in Stoynov (2012a) and Stoynov (2012b). Different application of ST(n,β) distribution are given in Stoynov (2011) and Kraynukov and Stoynov (2011, in Russian).

Stoynov process

We say that a process $X(t)$ is Stoynov process and denote this fact $X(t) \in ST(t;n,\beta),$ if it has the following properties:

1) $X(0)=0.$

2) The process has independent increments, i. e. for

$$0 < t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_1)-X(0), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1})$$

are independent.

3) The process is continuous stochastically (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

where the limit is taken in probability.

4) $X(t)$ is pure jump process with jumps at times $T_{i'}, i=1,2,\ldots$ and jump sizes $\Delta X(T_i)=1.$

5) The intervals between two jumps are

$$\tau = T_i - T_{i-1} \in ST(n-1,\beta), i=0,1,\ldots,T_0=0$$

The first three conditions mean that $ST(n,\beta)$ process is an additive process. If we replace condition 4) with condition

4') $X(t)$ is pure jump process with jumps at times $T_{i'}, i=1,2,\ldots$ and jump sizes $\Delta X(T_i)=Y_i,$ where Y_i are independent and identically distributed random variables, we say that $X(t)$ is compound $ST(n,\beta)$ process.

Simulation of Stoynov distribution

The simulation of $ST(n,\beta)$ distribution and corresponding PMF graphics for different values of the parameters n and β can be done by using R language. Here is the code for creating graphic representation of the PMF (for $n=100$ and $\beta=2$):

```
#ST(n,beta) graphics
beta<-2
n<-100
In_beta<-function(n,beta, recursive=TRUE)
{if (n==0) return (1/beta) else return (1/beta+n/beta*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
STn_beta<-function(x,n,beta){Cn_beta(n,beta)*(1+x)^n*exp(-x*beta)}
m<-10000
STn_beta_values<-0:m
T<-100
tvec<-T*0:m/m
for (i in 1:m) {
          STn_beta_values[i]<-STn_beta(i*T/m,n,beta)
}
freq<-10   # take out every 1000th point in the simulation & plot it
pick<-freq*(0:(m/freq))+1
title<-paste("ST(", round(n,1), ",",round(beta,1), ")")
plot(tvec[pick],STn_beta_values[pick],type='l',xlab="x",ylab="f(x)",main=title,ylim=c(
0,0.2))
```

The graphics for the case $n=100$ and $\beta=2$ is given in Fig 1. The graphics for the case $n=40$ and $\beta=2$ is given in Fig 2.

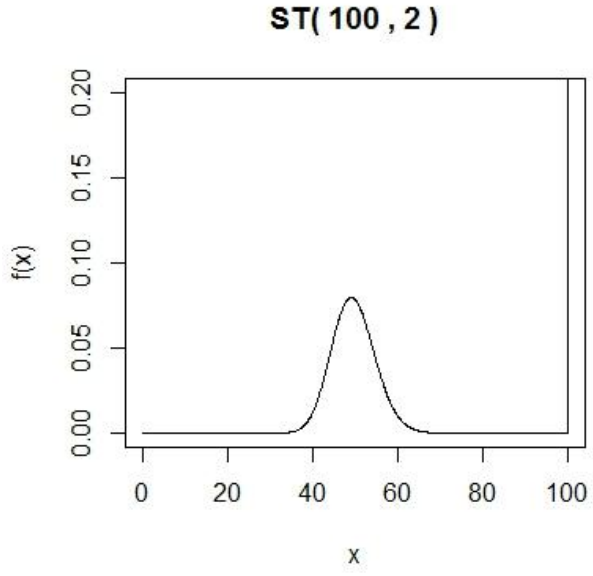


Figure 1. The graphics of $ST(n,\beta)$ distribution for the case $n=100$ and $\beta=2$.

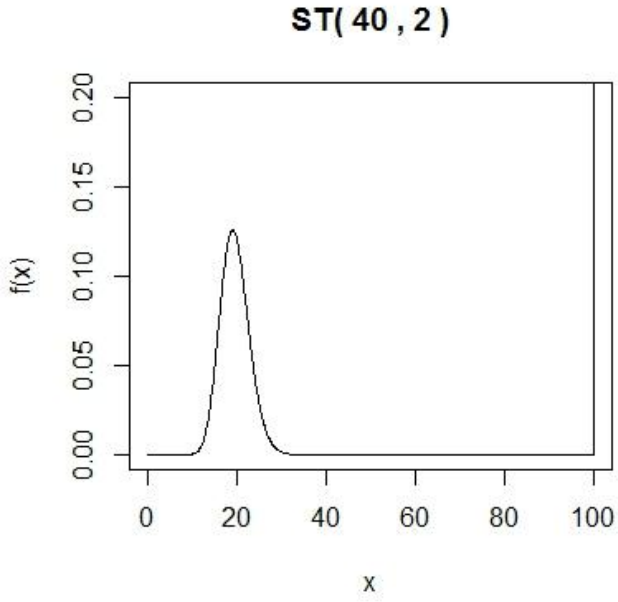


Figure 2. The graphics of $ST(n,\beta)$ distribution for the case $n=40$ and $\beta=2$.

Simulation of Stoynov process

To make simulation of Stoynov process, the following algorithm can be applied:

1. Define interval $[0,T]$ of the simulation.
2. Set $k=0.$
3. While $\sum_{i=1}^{k}\tau_i < T$ do:
   - 3.1. Set $k=k+1$
   - 3.2. Generate $\tau_k \in ST(n-1,\beta).$

3.3. Set $Y_k = 1$ for standard $ST(t;n,\beta)$ process or simulate Y based on a given distribution f for compound $X(t) \in ST(t;n,\beta)$ process.

Then the trajectory of $X(t) \in ST(t;n,\beta)$ is given by the formula

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where

$$N(t) = \sum_i 1_{\{T_i<T\}}.$$

Here is the code for creating graphic representation of the $ST(t;n,\beta)$ process:

```
#ST(n+1,beta)_process simulation
# Theta_grid_sets Poisson, Erlang, ST
T<-1000
lambda<-0.1
Theta_Poisson <-rexp(1,lambda)
while(max(Theta_Poisson)<T) Theta_Poisson <-c(Theta_Poisson,max(Theta_Poisson)+rexp(1,lambda))
m<-length(Theta_Poisson)
Theta_Poisson[m]<-T

# Erlang
beta_ER<-lambda
n_ER<-2
m_ER<-m/n_ER
Theta_ER<- Theta_Poisson[n_ER]
for (i in 2:m_ER)
{
Theta_ER<- c(Theta_ER, Theta_Poisson[i*n_ER])
}


#ST
ST_n<-5
ST_beta<-0.1
n<-20
tvec<-T*0:n/n
In_beta<-function(n,beta, recursive=TRUE)
{if (n==0) return (1/beta) else return ((1/beta)+(n/beta)*In_beta(n-1,beta))}
Cn_beta<-function(n,beta) {1/In_beta(n,beta)}
FDn_inverse<-function(y,n,beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,beta)*factorial(n)/(beta^i*factorial(n-i+1)))
}
Prob_acum<-Dn_prob[1]+Dn_prob[2]
for (k in 1:n)
{
if (y< Prob_acum) return (k)
else
Prob_acum<-Prob_acum+Dn_prob[k+2]
}
return(n+1)
}


#Dn_simulation<-function(n,ST_n,ST_beta){
Dn_values<-1:(n+1)
Dn_prob<-1:(n+1)
Dn_simulated<-1:n
for (i in 1:(n+1))
{
Dn_prob[i]<-(Cn_beta(n,ST_beta)*factorial(n)/(ST_beta^i*factorial(n-i+1)))
}
for (i in 1:n)
{
St_unif_value<- runif(1, min = 0, max = 1)
Dn_simulated[i]<-FDn_inverse(St_unif_value,ST_n,ST_beta)
}
acc_ST<- Dn_simulated[1]
```

```
Theta_ST<- Theta_Poisson[acc_ST]

while(acc_ST<m)
{
acc_ST<-(acc_ST+Dn_simulated[i])
if (acc_ST<=m)
Theta_ST<- c(Theta_ST, Theta_Poisson[acc_ST])
}
Theta_ST
s<-8000
tvec<-T*0:s/s
N<-0:s
N[1]<-0
u<-length(Theta_ST)

for (i in 1:s)
{
          N[i]<-0
          for (j in 1:u)
          {
                    if ((T*(i-1)/s)> Theta_ST[j]) N[i]<-(N[i]+1)
          }
}
title<-paste("Траектория на ST процес n=",round((ST_n+1),1), ", beta=", round(ST_beta,2))
plot(tvec[1:(s-1)],N[1:(s-1)],type='l', xlab="t",ylab="N(t)", main=title)
```

The graphics of a trajectory of $ST(n,\beta)$ process for the case $n=6$ and $\beta=0.1$ is given in Fig 3.

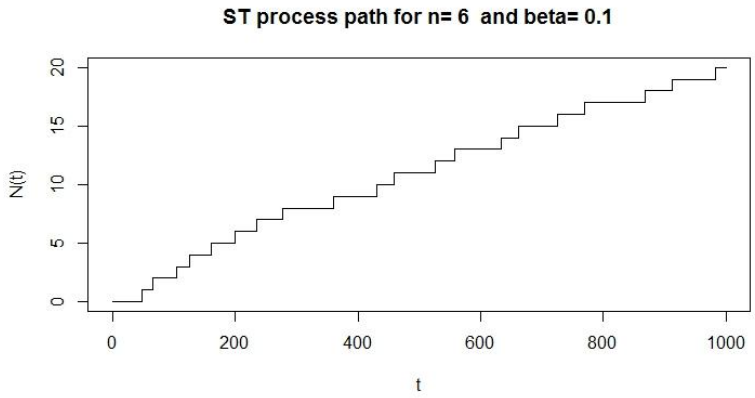


Figure 3. The graphics of a path of $ST(t;n,\beta)$ process for the case $n=6$ and $\beta=0.1$.

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